Quantum Theory Of Radiation

→ Heat transfers from one pt. to another by three modes:-
→ Conduction ⎤ Requires the presence of material medium filling
→ Convection ⎦ the space b/w the source & the receiver.
→ Radiation → Material medium is not necessary.
It can travel through vaccum too.

→ The energy radiated by a hot body is termed as radiant energy.

→ Heat radiations are EM in nature, like light.
→ The posn of heat radn in EM spectrum is beyond the red end of the spectrum i.e. infrared region having wavelengths ranging from 8×10^{-5} cm to 0.04 cm. (red colour to visible light)
→ The heat radn obey nearly all the laws of light, like rectilinear propogation, refln, refraction, interference, diffn, polarn etc.

Thermal Radiation

Acc. to maxwell, thermal radn is defined as the transfer of heat from a hot body to a cooler body without appreciable heating of the intervening medium (or space).
Thermal radn has the following properties:-
1) Thermal radn has EM wave nature & hence travels through empty space with vel. of light.
2) Like light, thermal radn travels in straight line.
3) Thermal radn obeys the law of inverse square.
4) It exhibits reflection, refraction interference, diffn & polarn phenomena.

PLANCK's QUANTUM POSTULATES

1) A black body rad^n chamber is filled up not only with rad^n, but also with simple harmonic oscillators or resonators of molecular dimensions known as Planck's oscillators which can vibrate with all possible frequencies.

2) The oscillators can't radiate or absorb energy ctsly, but energy is emitted or absorbed in the form of packets or quanta called photons.
Each photon has energy $= h\nu$, $h \rightarrow$ planck's const $= 6.625 \times 10^{-34}$ J-s.
$\nu \rightarrow$ freq of rad^n.
$\Rightarrow$ ~~Each~~ Exchange of energy b/w rad^n & matter can't take place ctsly but only in multiples of fundamental freq. of resonator.
$\therefore$ Energy emitted or absorbed $= 0, h\nu, 2h\nu, \ldots n h\nu$.

DERIVATION OF PLANCK's RADIATION LAW

In order to derive planck's rad^n law, we first derive the no. of resonators per unit vol. lying in the freq. range ν & $\nu + d\nu$ & then the avg. energy of planck's resonator.

(a) No. of resonators per unit vol. lying in the freq range ν & $\nu + d\nu$

→ Consider the rad^n to be enclosed in a hollow cubic encloser, the walls of which are perfectly reflecting.
→ Acc. to EM theory, the rad^n is supposed to consists of a no. of waves, travelling in all possible dir^{n}s in the encloser & suffer multiple reflections from the various walls of the encloser.
→ Inside the encloser, stationary waves form with the walls as nodal pts as a result of interference of reflected waves with the corresponding incident waves. This is just like the vibr^{n}s of stretched string with both end pts fixed.

→ With this analogy, it is well known that the nodes are formed at the end pts of the string due to stationary vibr^{n}s & only certain discrete frequencies of vibr^{n}s are allowed.

→ If l is the length of the string, allowed wavelengths are,

$$\lambda = \frac{2l}{n} \; ; \; n = 1, 2, \ldots \infty \quad \text{—} (1)$$

Allowed freqs :- $\nu = \frac{c}{\lambda} = \frac{nc}{2l} \; ; \; n = 1, 2, \ldots \infty$

where $c \rightarrow$ speed of waves.

→ Every allowed freq is called a mode of vibr^n.

→ Allowed modes of vibr^n inside a cube can be calculated just as in the case of string. But in this case, the waves are confined to a 3-D space.

→ Let each side of a cube be a. If we take three intersecting edges of the cube as X, Y, & Z-axes of a co-ordinate system.

→ $\cos\alpha$, $\cos\beta$ & $\cos\gamma$ are the dir^n cosines of propogation of particular wave, then the projns of the edges of the cube on the dir^n of propogation of wave are $a\cos\alpha$, $a\cos\beta$, & $a\cos\gamma$.

→ In this case, only the waves will be allowed for which all the faces of the cube form nodal planes. For this, the allowed wave length have to satisfy 3 cond^{n}s of kind of eq^n (1) in which l is replaced by $a\cos\alpha$, $a\cos\beta$ & $a\cos\gamma$ respectively.

∴ Allowed wavelength λ must satisfy

$$\lambda = \frac{2a\cos\alpha}{n_1}, \quad \lambda = \frac{2a\cos\beta}{n_2}, \quad \lambda = \frac{2a\cos\gamma}{n_3} \quad \text{—} (2)$$

where n_1, n_2, n_3 are +ve integers.

Using rel^n, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ — (3)

$$\frac{n_1^2 d^2}{4a^2} + \frac{n_2^2 d^2}{4a^2} + \frac{n_3^2 d^2}{4a^2} = 1$$

$$\Rightarrow \quad n_1^2 + n_2^2 + n_3^2 = \frac{4a^2}{d^2} = \left(\frac{2a}{d}\right)^2 = \left(\frac{2a\nu}{c}\right)^2 \quad \text{— (4)}$$

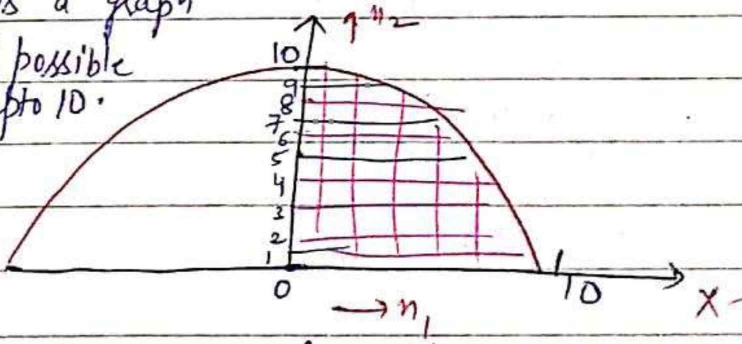
→ This eq^n gives the allowed freq (or modes of vibr^n) inside a cube.
→ The total no. of modes of vibr^n are total no. of possible sets (n_1, n_2, n_3).

∴ The no. of modes of vibr^{ns} within the freq interval ν & $\nu + d\nu$ can be calculated using eq^n (4).

* Let us first count the modes of vibr^n in an analogous 2-D problem

2-D eq^n :- $n_1^2 + n_2^2 = \left(\frac{2a\nu}{c}\right)^2$. → This reprs. a circle of radius $\left(\frac{2a\nu}{c}\right)$.

Along x-axis → n_1
y-axis n_2.
} This fig shows a graph having n_1 & n_2 all possible integral values upto 10.



→ The pts on the circle corresponds to the freq ν while those lying inside the circle less than ν.
→ The lines drawn divide the quadrant in a no. of unit squares.
→ every pt of intersection can be associated with one square.
Area of each square is ~~one~~ unit.

$\therefore$ No. of squares = Area inside the quadrant of the circle.

$\Rightarrow$ No. of modes of vibr^n within the freq interval ν & $\nu + d\nu$.
= Area ~~of~~ in the +ve quadrant lying b/w two circles of radii $\frac{2a\nu}{c}$ & $2a\left(\frac{\nu+d\nu}{c}\right)^2$.

$$\therefore \text{Area} = \frac{\pi}{4}\left[\left\{2a\left(\frac{\nu+d\nu}{c}\right)\right\}^2 - \left\{\frac{2a\nu}{c}\right\}^2\right]$$

$$= \frac{\pi}{\cancel{4}} \times \frac{\cancel{4}a^2}{c^2}\left\{(\nu+d\nu-\nu)(\nu+d\nu+\nu)\right\}$$

$$= \frac{\pi a^2}{c^2} \times d\nu \cdot (2\nu + d\nu)$$

$$\Rightarrow \frac{\pi a^2}{c^2} d\nu \times 2\nu = \frac{2\pi a^2 \nu d\nu}{c^2}.$$

$\therefore$ In 3-D, No. of modes of vibr^n. in the freq range ν & $\nu + d\nu$
= vol. of Const ($\frac{1}{8}$th) of the spherical shells with radii $\frac{2a\nu}{c}$ & $2a\frac{(\nu+d\nu)}{c}$.

$$= \frac{1}{8}\left[\frac{4\pi}{3}\left(\frac{2a\nu}{c}\right)^3 - \frac{4\pi}{3}\left\{\frac{2a(\nu+d\nu)}{c}\right\}^3\right]$$

$$= \frac{1}{8} \times \left(\frac{4}{3}\pi\right) \times \frac{8a^3}{c^3}\left[\cancel{\nu^3} - \cancel{\nu^3} + 3\nu^2 d\nu\right]$$

$$= \frac{4\pi a^3 \nu^2}{c^3} d\nu.$$ $a^3 = V$ = vol. of cube.

$\therefore$ No. of modes of vibr^n inside cubical enclosed with freq range ν & $\nu + d\nu$ $= \frac{4\pi V \nu^2}{c^3} d\nu$.

$\therefore$ No. of modes of vibrⁿ per unit vol. in the freq range ν & $\nu+d\nu$

$$= \frac{4\pi\nu^2 d\nu}{c^3}$$

Note :- Since the black body radⁿ travel with vel. of light c & are tranverse in characters ~~hence modes of vibrⁿ are double as that of longitudnal waves~~. unlike sound waves in the string which are longitudnal.

→ As there are two possible polarisations for each tranverse wave, the modes vibrⁿ of transverse wave is double as for longitudnal waves.

$\therefore$ For Black body radⁿ or EM waves, the no. of modes of vibrⁿ per unit vol. within the freq range ν & $\nu+d\nu$:-

$$= 2 \times \frac{4\pi\nu^2 d\nu}{c^3} = \frac{8\pi\nu^2 d\nu}{c^3}.$$

Average Energy of Planck's Oscillator :—

→ If N is the total no. of planck's resonators & E their total energy then average energy per Planck's oscillator :-

$$\bar{\epsilon} = \frac{E}{N}.$$

~~→ Acc. to Maxwell's law of molecular motion, if ϵ is a energies 0, ϵ, 2ϵ, ---- $r\epsilon$, -- are in a ratio, 1 : $e^{-\epsilon/KT}$: $e^{-2\epsilon/KT}$~~

Acc. to maxwell, P that a resonator will posses engy ϵ is $e^{-\epsilon/KT}$

Let N_0 is the no. of resonator having zero engy

N_1 " " " " " ϵ " " $\quad N_1 = N_0 \exp(-\epsilon/K$

N_2 " " " " " 2ϵ $\quad N_2 = N_0 \exp(-2\epsilon$

$\therefore$ Total no. of resonators, $N = N_0 + N_1 + N_2 + \cdots + N_r + \cdots$

$$= N_0 + N_0 e^{-\epsilon/kT} + N_0 e^{-2\epsilon/kT} + \cdots e^{-r\epsilon/kT} + \cdots$$

Let $e^{-\epsilon/kT} = y$

$$\therefore \quad N = N_0 \left[1 + y + y^2 + \cdots + y^r + \cdots\right]$$

$$N = \frac{N_0}{1-y}.$$

$\therefore$ Total energy of Planck's oscillator,

$$E = 0 \times N_0 + \epsilon \times N_1 + 2\epsilon \times N_2 + \cdots + r\epsilon \times N_r + \cdots$$

$$= N_0 \epsilon \left[1 \cdot e^{-\epsilon/kT} + \cdots + r e^{-r\epsilon/kT}\right]$$

$$= N_0 \epsilon \, e^{-\epsilon/kT} \left[1 + 2y + 3y^2 + \cdots r \cdots\right]$$

$$= N_0 \epsilon \cdot y \frac{1}{(1-y)^2}$$

$$\bar{\epsilon} = \frac{E}{N} = \frac{\cancel{N_0} \epsilon \cdot y}{(1-y)^2} \times \frac{(1-y)}{\cancel{N_0}}$$

$$= \frac{\epsilon y}{1-y} = \frac{\epsilon \cdot e^{-\epsilon/kT}}{1 - e^{-\epsilon/kT}}$$

$$= \frac{\epsilon}{e^{\epsilon/kT} - 1}.$$

Scanned with CamScanner

Acc to Planck's hypothesis of quantum theory, $\epsilon = h\nu$

$\therefore$ Avg Energy of Planck's oscillator, $\bar{\epsilon} = \frac{h\nu}{e^{h\nu/kT}-1}$

$\because$ Energy density belonging to the range $d\nu$;

$$E_\nu d\nu = \underbrace{\left(\frac{8\pi\nu^2 d\nu}{c^3}\right)}_{N} \cdot \underbrace{\left(\frac{h\nu}{e^{h\nu/kT}-1}\right)}_{\bar{\epsilon}}$$

$$\boxed{E_\nu d\nu = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/kT}-1} d\nu.}$$

$\rightarrow$ PLANCK'S RADIATION LAW

Where $E_\nu d\nu$ is energy density (total energy per unit vol) belonging to the range $d\nu$

Energy density $E_\lambda d\lambda$ belonging to range $d\lambda$ can be obtained by using $\nu = \frac{c}{\lambda}$ & $|d\nu| = \left|-\frac{c}{\lambda^2} d\lambda\right|$

$$E_\lambda d\lambda = \frac{8\pi h}{c^3}\left(\frac{c^3}{\lambda^3}\right)\frac{1}{e^{hc/\lambda kT}-1} \cdot \left|\frac{-c}{\lambda^2} d\lambda\right|$$

$$\boxed{E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{e^{hc/\lambda kT}-1}}$$

$\rightarrow$ Planck's Radn Law in terms of λ.

WEIN'S LAW FROM PLANCK'S LAW FOR SHORTER WAVELENGTHS

Planck's rad^n law, $E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{e^{hc/\lambda kT} - 1}$

For shorter wavelengths, $e^{hc/\lambda kT}$ becomes large as compared to unity & hence planck's law reduces to :-

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{e^{hc/\lambda kT}}$$

$$\boxed{E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} e^{-hc/\lambda kT} \cdot d\lambda}$$ → WEIN'S LAW.

RAYLEIGH-JEANS LAW (FOR LONGER WAVELENGTHS)

For longer wavelengths, $e^{hc/\lambda kT} \approx 1 + \frac{hc}{\lambda kT}$

$$\therefore E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{\cancel{1} + \frac{hc}{\lambda kT} - \cancel{1}}$$

$$= \frac{8\pi hc}{\lambda^5} \cdot \frac{\lambda kT}{hc} d\lambda$$

$$\boxed{E_\lambda d\lambda = \frac{8\pi kT}{\lambda^4} \cdot d\lambda}$$ → Rayleigh Jeans law

WEIN'S DISPLACEMENT LAW

Planck's law, $E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT}-1}$ — ①

Differentiating eqⁿ ① w.r.t. λ partially,

$$\frac{dE_\lambda}{d\lambda} = \frac{1}{e^{hc/\lambda kT}-1} \times \frac{(-5)(8\pi hc)}{\lambda^6} + \frac{8\pi hc}{\lambda^5} \times \frac{\frac{hc}{\lambda^2 kT} \cdot e^{hc/\lambda kT}}{\left(e^{hc/\lambda kT}-1\right)^2}$$

$$= \frac{-40\pi hc}{\lambda^6} \times \frac{1}{e^{hc/\lambda kT}-1} + \frac{8\pi hc}{\lambda^5} \times \frac{hc}{\lambda^2 kT} \cdot \frac{e^{hc/\lambda kT}}{\left(e^{hc/\lambda kT}-1\right)^2}$$

for maximum energy of emission i.e. for max. value of E_λ, $\frac{dE_\lambda}{d\lambda} = 0$

$$\Rightarrow \frac{-40\pi hc}{\lambda^6} \cdot \frac{1}{e^{hc/\lambda kT}-1} + \frac{8\pi hc}{\lambda^5} \times \frac{hc}{\lambda^2 kT} \times \frac{e^{hc/\lambda kT}}{\left(e^{hc/\lambda kT}-1\right)^2} = 0$$

$$\Rightarrow \frac{8\pi hc}{\left(e^{hc/\lambda kT}-1\right)\lambda^6} \left[-5 + \frac{hc}{\lambda kT} \times \frac{e^{hc/\lambda kT}}{\left(e^{hc/\lambda kT}-1\right)} \right] = 0$$

Let $\frac{hc}{\lambda kT} = x$

$$\therefore -5 + \frac{x e^x}{e^x - 1} = 0$$

$$\Rightarrow \frac{x e^x}{e^x - 1} = 5$$

$$x = \frac{hc}{\lambda kT} = 4.965$$

Scanned with CamScanner

Here λ represents the wavelength at which energy per unit range of wavelength is max.
Representing this particular wavelength by λ_m,

$$\frac{hc}{\lambda_m kT} = 4.965$$

$$\Rightarrow \lambda_m T = \frac{hc}{4.965\,K} = \text{constt}$$

$$\Rightarrow \boxed{\lambda_m T = \text{constt}} \rightarrow$$ Wein's displacement Law.

STEFAN's LAW

Planck's rad^n law in terms of freq.

$$E_\nu \, d\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/KT} - 1} \, d\nu$$

∴ Total radiant energy over all frequencies,

$$E = \int_0^\infty E_\nu \, d\nu = \frac{8\pi h}{c^3} \int_0^\infty \frac{\nu^3 \, d\nu}{e^{h\nu/KT} - 1}$$

Put $\frac{h\nu}{KT} = x \Rightarrow \nu = \frac{KT}{h} x \Rightarrow d\nu = \frac{KT}{h} dx$

$$\therefore E = \int_0^\infty E_\nu \, d\nu = \frac{8\pi K^4 T^4}{c^3 h^3} \int_0^\infty \frac{x^3 \, dx}{e^x - 1}$$

$$\int_0^{\infty} \frac{x^3\,dx}{e^x - 1} = \frac{\pi^4}{15}.$$

$$\therefore \quad E = \frac{8\pi^5 k^4 T^4}{15 c^3 h^3}$$

$$\Rightarrow E = A T^4 \quad \text{where } A = \frac{8\pi^5 k^4}{15 c^3 h^3}$$

But $\frac{Ac}{4} = \sigma$, $\rightarrow$ Stefan's constt.

$$\therefore \quad \boxed{E = \sigma T^4} \rightarrow \text{Stefan's Law}$$

where $\sigma = \frac{8\pi^5 k^4}{15 h^3 c^2}$