Sylow's theorem:-

Def^n:- Let G be a group and let p be a prime.

(1) A group of order p^α for some $\alpha \geq 0$ is called p-group. Subgroups of G which are p-groups are called p-subgroups.

(2) If G is a group of order $p^\alpha m$, where $p \nmid m$, then a subgroup of order p^α is called a Sylow-p-subgroup of G.

(3) The set of Sylow p-subgroups of G will be denoted by $Syl_p(G)$ and the number of Sylow-p-subgroups of G will be denoted by $n_p(G)$.

Examples

1. $\mathbb{Z}_9$. As $|\mathbb{Z}_9| = 9 = 3^2$

   $\therefore \mathbb{Z}_9$ is a p-group (3-group)

2. S_3, Let $H_1 = \langle (1\,2) \rangle$

   $H_2 = \langle (1\,2\,3) \rangle$

   then H_1, H_2 are p-subgp of G.

3. S_3, Let $H_1 = \langle (1\,2) \rangle$

   $H_2 = \langle (1\,2\,3) \rangle$

   then H_1 is Sylow-2-subgp & H_2 is Sylow 3-subgp of G.

(4) $\mathbb{Z}_{18}$, $|\mathbb{Z}_{18}| = 3^2 \times 2$

then $H_1 = \langle 2 \rangle$ & $H_2 = \langle 9 \rangle$

H_1 is sylow-3 subgp and H_2 is sylow-2 subgp.

and $|H_1| = 9$ & $|H_2| = 2$.

Also $n_3(\mathbb{Z}_{18}) = 1$ and $n_2(\mathbb{Z}_{18}) = 1$

Thm:- (sylow's theorem)

Let G be a group of order $p^\alpha m$, where p is a prime not dividing m. Then

(1) Sylow p-subgroups of G exist, i.e. $Syl_p(G) \neq \phi$

(2) If P is a ~~subgroup~~ sylow p-subgroup of G and Q is any p-subgroup of G, then there exists $g \in G$ such that $Q \leq gPg^{-1}$, i.e. Q is contained in some conjugate of P.

In particular, any two sylow p-subgroups of G are conjugate in G.

(3) The number of sylow p-subgroups of G is in the form $1+kp$ i.e.

$$n_p \equiv 1 \pmod{p}$$

Further, n_p is the index in G of the normalizer $N_G(P)$ for any sylow p-subgroup

P, hence n_p divides m.

i.e. $n_p = |G : N_G(P)|$, where P- Sylow p-subgp.

Examples:-

① S_3 $|S_3| = 6 = 2 \times 3$

$\therefore$ S_3 have Sylow 2-subgp and Sylow 3-subgps of order 2 and 3 respectively.

S_3 has only one subgp of order 3 i.e. $\langle(1\,2\,3)\rangle$

$\therefore$ $n_3(S_3) = 1$.

and S_3 has 3 subgps of order 2

$\therefore$ $n_2(S_3) = 3$

and all subgps of order 2 are conjugate to each other.

Corollary (20):- Let P be a Sylow p-subgroup of G. $|G| = p^\alpha m$.

Then the following are equivalent.

(1) P is unique Sylow p-subgp of G, i.e. $n_p = 1$.

(2) P is normal in G.

(3) P is characteristic in G.

(4) All subgroups generated by elements of p-power order are p-groups, i.e. if X

is any subset of G such that $|x|$ is a power of p for all $x \in X$, then $\langle X \rangle$ is a p-group.

Proof:- We will prove the result in following way.

$$(4) \Leftrightarrow (1) \Leftrightarrow (2) \Leftrightarrow (3).$$

Let (1) holds, i.e. P is unique Sylow p-subgp of G

then $gPg^{-1} = P$ for all $g \in G$ (Sylow's 2nd thm)

$\Rightarrow$ $P \trianglelefteq G$.

$\Rightarrow$ (2) holds

Conversely let (2) holds, i.e. $P \trianglelefteq G$.

Let $P, Q \in Syl_p(G)$, then there exist

$g \in G$ s.t. $gPg^{-1} = Q$ (Sylow's 2nd thm)

But $gPg^{-1} = P$

$\therefore$ $P = Q$

$\Rightarrow$ P is unique Sylow p-subgp of G.

$\therefore$ $(1) \Leftrightarrow (2)$.

Let (2) holds, i.e. P is normal in G.

then P is unique Sylow p-subgp

$\Rightarrow$ P is unique subgp of G of order p^{α}.

and $|\phi(P)| = |P| \quad \forall \ \phi \in Aut(G)$

$\therefore \quad \phi(P) = P \quad \forall \ \phi \in Aut(G)$

$\Rightarrow$ P is characteristic in G.

Conversely, let P is characteristic in G

then P is normal in G ($\because$ Every characteristic subgp is normal)

$\therefore \quad (2) \iff (3)$

Now let (1) holds, we will show (4)

let X is a subset of G such that $|x|$ is power of p for all $x \in X$

$\Rightarrow \quad \langle x \rangle$ is a p-subgroup of G

Then By Sylow's 2nd theorem, $\exists \ g \in G$ s.t.

$$\langle x \rangle \le gPg^{-1} = P$$

$\Rightarrow \quad x \in gPg^{-1} = P$

$\Rightarrow \quad x \in P \quad \forall \ x \in X$

$\Rightarrow \quad X \subseteq P$

$\Rightarrow \quad \langle X \rangle \le P$

$\Rightarrow \quad \langle X \rangle$ is a p-group.

Conversely let (4) holds.

let X be the union of all Sylow p-subgroups of G.

then X is a p-group. (Infact X is a sylow p-subgroup of G).

If $P \in Syl_p(G)$, then $P \leq \langle X \rangle$

and since P is a p-subgroup of G of maximal order,

$\therefore \quad P = \langle X \rangle$.

$\Rightarrow$ P is unique.

$\therefore \quad (1) \Leftrightarrow (4)$. ∎

Examples:-

(1) Let G be a finite group and p be a prime. If p does not divide $|G|$, then sylow p-subgp of G is the trivial group.

If $|G| = p^{\alpha}$, then G is the unique Sylow p-subgp of G.

(2) Let G be a finite abelian group. Then it has unique Sylow p-subgroup for each prime p and this subgroup consists of all elements x whose order is a power of p.

(3) A_4, $|A_4| = 12 = 2^2 \times 3$

A_4 has sylow 2-subgp of order 4 and has sylow 3-subgp of order 3.

$\{(1), (12)(34), (13)(24), (14)(23)\}$ is unique sylow 2-subgp.

and $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$, $\langle(234)\rangle$ are sylow 3-subgps of A_4.

(4) S_4, $|S_4| = 24 = 2^3 \times 3$

$\Rightarrow$ S_4 has sylow 2-subgp of order 8 and has sylow 3-subgp of order 3.

and $n_2 = 3$ and $n_3 = 4$.

$H_1 = \{(1), (1\,2\,3\,4), (13)(24), (1\,4\,3\,2);$
$(13), (24), (14)(23), (12)(34)\}$

then H_1 is sylow 2-subgp

and $H_1 \cong D_8$.

$\Rightarrow$ Every sylow 2-subgp of S_4 is isomorphic to D_8.

Applications of Sylow's theorem:-

Def^n:- Simple group:- A group G is said to be simple group if it does not have any non-trivial normal subgroup.

Ex ① Let $G = \mathbb{Z}_8$, then G is not simple.

② Let $G = S_3$, then also S_3 is not simple.

In this section we'll use sylow's theorem to prove that a group of particular order is not simple.

Groups of order pq, p and q are prime and $p<q$

Let G be a group and $|G| = pq$ where p and q are primes and $p<q$. Let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$

Claim ①:- Q is normal in G.

As $n_q = 1 + kq$ for some $k \geq 0$

and n_q divides p.

$\Rightarrow$ $1+kq$ divides p

$\therefore$ $k=0$ $\Rightarrow$ $n_q = 1$.

$\therefore$ Q is normal in G.

Claim ②:- If P is normal in G, then G is cyclic.

let $P = \langle x \rangle$ and $Q = \langle y \rangle$ ($\because$ P & Q are cyclic having order p & q resp.)

We first show that $xy = yx$.

Consider $x^{-1}y^{-1}xy$, $x^{-1} \in P$

and $y^{-1}xy \in P$ ($\because P \trianglelefteq G$)

$\therefore$ $x^{-1}y^{-1}xy \in P$

Also, $x^{-1}y^{-1}x \in Q$ and $y \in Q$

$\Rightarrow$ $x^{-1}y^{-1}xy \in Q$

$\therefore$ $x^{-1}y^{-1}xy \in P \cap Q$

But $P \cap Q = 1$

$\therefore$ $x^{-1}y^{-1}xy = 1$

$\Rightarrow$ $xy = yx$.

And As $|xy| = \text{lcm}(|x|, |y|) = \text{lcm}(p, q)$

$= pq$.

$\therefore$ $G = \langle xy \rangle$.

$\Rightarrow$ G is cyclic.

Remarks:- Let $|G| = pq$, if $p \nmid q-1$ then G is cyclic.

Groups of Order 30

Let G be a group of order 30. Let $P \in Syl_5(G)$ and $Q \in Syl_3(G)$

Claim:- G has a normal subgroup isomorphic to $\mathbb{Z}_{15}$. (or G has normal cyclic subgroup)

first we will show that either P or Q is normal.
for that assume that none of P and Q are normal.
Then n_5, the no. of Sylow 5-subgroups in G is of the form $1+5k$. and n_5 divides 6

$\therefore$ n_5 can be 1 or 6.

$\Rightarrow$ $n_5 = 6$ $\left(\because n_5 \neq 1 \text{ if no then } P \trianglelefteq G \right)$

Similarly, we get that $n_3 = 10$.

As each element of order 5 lies in a Sylow 5-subgp and each Sylow 5-subgp contains 4 non identity elements having order 5.

Also let $P_1, P_2 \in Syl_5(G)$

Then either $P_1 \cap P_2 = P_1$ or $P_1 \cap P_2 = 1$.

$\therefore$ No. of elements of order 5 in G $= 4 \times 6$
$= 24$. — (1)

Also, as each element of order 3 lies in a Sylow 3-subgp and each Sylow 3-subgp contains 2 non-identity elements having order 3.

Also if $Q_1, Q_2 \in Syl_3(G)$

Then either $Q_1 \cap Q_2 = Q_1$ or $Q_1 \cap Q_2 = 1$

$\therefore$ No. of elements of order 3 in G $= 2 \times 10 = 20$ — (2)

From ① and ②, we get that G has atleast 44 elements which contradicts the fact that $|G| = 30$. Therefore our assumption is wrong.

Hence either P or Q is normal in G.

Then PQ is a group of order 15.

And $|G : PQ| = 2$

Then By Index theorem

$$PQ \trianglelefteq G.$$

And also By remarks PQ is cyclic.

$$\therefore PQ \cong \mathbb{Z}_{15}.$$

Hence G has a normal subgroup isomorphic to $\mathbb{Z}_{15}$.

<u>Results</u> :-

1.) Let G be a group with subgroups H and K with $H \leq K$. Then if H is characteristic in K and K is normal in G, Then $H \trianglelefteq G$.

2.) Let G be a group with subgroups H and K with $H \leq K$. Then if H is characteristic in K and K is characteristic in G, then H is characteristic in G.

Groups of order 12

Let G be group and $|G| = 12$. Then G has Sylow 2-subgp of order 4 and Sylow 3-subgp of order 3.

Claim:- Either Sylow 3-subgp is normal or $G \cong A_4$ (In this case Sylow 2-subgp is normal).

Proof:- Let $P \in Syl_3(G)$ and assume that P is not normal in G.

$\therefore \quad n_3 \neq 1$

Since $n_3 | 4$ and $n_3 = 1 + 3k$

Therefore $n_3 = 4$

$\Rightarrow$ G has 4 sylow 3-subgps

$\Rightarrow$ No. of subgps conjugate to P is 4

Also, No. of subgps conjugate to $P = |G : N_G(P)|$

$$\Rightarrow \quad |G : N_G(P)| = 4$$

$$\Rightarrow \quad |N_G(P)| = 3.$$

And $P \leq N_G(P) \Rightarrow N_G(P) = P$.

Also, G has 8 elements of order 3.

Let A be the set of Sylow 3 subgps of G and G acts on A by conjugation.

Then this action affords permutation representation

$$\phi : G \to S_A$$

and $K = \ker\phi = \{ g \in G \mid g \cdot a = a \ \forall a \in A \}$

$$= \{ g \in G \mid gag^{-1} = a \ \forall a \in A \}$$

$$\Rightarrow \quad K \leq N_G(a) \ \forall a \in A$$

In particular $\quad K \leq N_G(P)$

As $\quad K \trianglelefteq G \quad \Rightarrow \quad K \neq N_G(P)$

$$\Rightarrow \quad |K| = 1 .$$

Then by 1^{st} isomorphism thm.

$$\frac{G}{K} \cong \phi(G) \leq S_A$$

$$\Rightarrow \quad G \cong \phi(G) \leq S_A$$

and as $|A| = 4$, $\therefore \ S_A \cong S_4$

Hence $\quad G \cong G_1 \leq S_4$

As G has 8 elements of order 3, Therefore G_1 has 8 elements of order 3. Also S_4 has 8 elements of order 3 which are contained in A_4.

$$\therefore \quad |G_1 \cap A_4| \geq 8 \quad \Rightarrow \quad G_1 = A_4$$

$\therefore\ G \cong A_4$.

And as A_4 has normal sylow 2-subgp

$\therefore$ G has normal sylow 2-subgp.

Groups of order p^2q, p and q are distinct primes

let G be a group and $|G| = p^2q$.

Claim:- G has a normal sylow subgp.

Proof: let $P \in Syl_p(G)$ and $Q \in Syl_q(G)$.

consider first case when $p > q$

then $n_p = 1 + kp$ and n_p divides q

$\Rightarrow n_p = 1$

$\Rightarrow P \trianglelefteq G$.

Now consider the case when $p < q$

then $n_q = 1 + qk$ and n_q divides p^2.

Possible values of n_q are $1, p$ or p^2.

If $n_q = 1$, then we are done

Now suppose $n_q \neq 1 \Rightarrow n_q = p$ or p^2.

As $q > p$ we cannot have $n_q = p$.

$\therefore\ n_q = p^2$

$\Rightarrow \quad 1 + kq = p^2$

$\Rightarrow \quad kq = p^2 - 1 = (p-1)(p+1)$

Then either $q \mid p-1$ or $q \mid p+1$

And $q \nmid p-1$ as ~~p≥q~~ $q > p$

Therefore $q \mid p+1$ and $q > p$

$\Rightarrow \quad q = p+1$

$\Rightarrow \quad p = 2$ and $q = 3$

$\Rightarrow \quad |G| = 12$ and G has a normal sylow subgp.

<u>Groups of order 60</u>

<u>Prop^n 21</u>:- If $|G| = 60$ and G has more than one sylow 5-subgp, then G is simple.

<u>Proof</u>:- We will prove it by the way of contradiction. Suppose G is not simple and G has more more than one sylow-subgp.

Let H be non-trivial normal subgp of G.

and $n_5 = 1 + 5k$ and n_5 divides 12

$\Rightarrow \quad n_5$ can be 1 or 6

But $n_5 \neq 1 \Rightarrow n_5 = 6$

If 5 divides $|H|$ then H contains a sylow 5 subgp of G and since H is normal.

Therefore H contains all the conjugacy class of sylow 5-subgp

$\Rightarrow$ H contains all the 6 sylow 5-subgps of G.

$\Rightarrow$ H contains 24 elements of order 5

$\therefore$ $|H| \geq 1+24 = 25$

and as $H \leq G \Rightarrow |H| \,\big|\, |G|$

$\therefore$ $|H| = 30$

$\Rightarrow$ H has a normal sylow 5-subgp ($\because$ By previous result of $|G| = 30$)

which is a contradiction.

$\therefore$ 5 does not divides $|H|$

Now the possibilities of $|H|$ are 2, 3, 4, 6 or 12.

Now assume $|H| = 2, 3$ or 4.

let $\bar{G} = \frac{G}{H}$

then $|\bar{G}| = 30, 20$ or 15 respectively.

In each cases $\bar{G}$ has a normal sylow 5-subgp.

let $\bar{P}$ is normal sylow 5-subgp of $\bar{G}$.

Let $\phi : G \to \bar{G}$ be natural mapping

i.e. $\phi(g) = \bar{g} = gH$

Then ϕ is homomorphism.

As $\bar{P} \trianglelefteq \bar{G}$, then $\phi^{-1}(\bar{P}) \trianglelefteq G$

Let $H_1 = \phi^{-1}(\bar{P})$

then $H_1 \trianglelefteq G$ and $H_1 \neq 1$ or G

and as $|P| \,\big|\, |H_1| \Rightarrow 5 \,\big|\, |H_1|$

which contradicts the fact that 5 cannot divide any non-trivial normal subgp of G's order.

$\therefore$ $|H| \neq 2, 3$ or 4.

Now possibilities of $|H|$ is 6 and 12.

Let $|H| = 6$ or 12.

Then H has normal sylow subgps.

$\Rightarrow$ H has characteristic subgps (Corollary no).

Let K be normal sylow subgp of H

then K is characteristic subgp of H ~~wh~~

and H is normal subgp of G.

$\therefore$ $K \trianglelefteq G$ (Result 1)

and $|K| = 2, 3$ or 4

which contradicts the fact that any non trivial normal subgp of G cannot have order 2, 3 or 4.

$\therefore$ $|H| \neq 6$ or $|H| \neq 12$.

This contradicts the fact that H is non-trivial.

$\therefore$ G is simple.

Corollary 22 :- A_5 is simple

Proof:- $|A_5| = \frac{5!}{2} = 60$

and $\langle(1\,2\,3\,4\,5)\rangle$ and $\langle(1\,3\,2\,4\,5)\rangle$ are two distinct sylow 5-subgps of A_5.

$\therefore$ By ~~Corollary 21~~ propn 21, A_5 is simple.

Simple Groups:-

A group is simple if its only normal subgroups are the identity subgroup and the group itself.

Example:-

Let G be finite abelian group.

then, if $|G| = p$, where p is prime

then G is simple.

and if $|G| = p_1 p_2 \cdots p_n$, p_i are prime

then G is not simple.

So, we will see the simplicity of finite groups non-abelian groups.

* A_5 is only simple gp whose order is less than 168.

Thm:- (25.1) Sylow Test for non-simplicity

Let n be a positive integer that is not prime, and let p be a prime divisor of n. If 1 is the only divisor of n that is congruent to 1 modulo p, then there does not exist a simple group of order n.

Proof: Let $|G| = n$

Case 1 if $n = p^r$.

$\Rightarrow$ $Z(G)$ is non-trivial. (Thm 8 of sec 4.3 in Dummit)

And $Z(G) \trianglelefteq G$, hence G is not simple.

Case: if n is not a prime-power, let $|G| = p^r m$, $p \nmid m$

then as p divides n.

G has sylow p subgp. Let $P \in Syl_p(G)$

and $n_p = 1 + kp / m$

$\Rightarrow n_p = 1$ (Given, as 1 is only divisor of n that is congruent to 1 modulo p).

$\Rightarrow$ P is normal.

and $|P| = p^r \leq |G| \Rightarrow P$ is non-trivial.

$\Rightarrow$ G is not simple.

Examples ① Let $|G| = 280 = 2^3 \cdot 5 \cdot 7$

then ~~5 divides $|G|$.~~

and ~~1 is only~~ divisor of 280 that is congruent

to 1 modulo 5.

i.e. if $1+5k \mid 280 \Rightarrow k = 0$.

① $|G| = 21 = 3 \times 7$

then 7 divides $|G|$.

and if $1+7k \mid 21 \Rightarrow k = 0$

$\Rightarrow$ G is simple.

② $|G| = 70 = 2 \times 5 \times 7$

$1+7k \mid 70 \Rightarrow k = 0$

$\Rightarrow$ G is simple.

Thm 25.2 2. Odd test

An integer of the form $2n$, where n is odd number greater than 1, is not the order of a simple group.

Proof: Let G acts on itself by left multiplication then the associated permutation representation

$$\phi : G \longrightarrow S_G \approx S_{2n}$$

and kernel of this action is $\{e\}$

$$\therefore \quad G \approx \phi(G) \leq S_{2n}$$

As $|G| = 2n$

By Cauchy theorem, $\exists\, x \in G$, s.t. $|x| = 2$

then $|\phi(x)| = 2$ and $\phi(x) \in \phi(G) \leq S_{2n}$.

$\Rightarrow$ $\phi(x)$ in disjoint cycle form and each cycle must have length 1 or 2.

But $\phi(x)$ does not contain 1 cycles, as

$$\phi(x) = \sigma_x = gx \quad \forall\, g \in G.$$

Now let $\sigma_x(g') = g'$

$$\Rightarrow \quad xg' = g' \quad \Rightarrow \quad x = e, \text{ contradiction}$$

$$\Rightarrow \quad \sigma_x(g') \neq g \quad \forall\, g \in G.$$

$\Rightarrow$ $\phi(x) = \sigma_x$ does not contain 1 cycle

Thus, in cycle form, $\phi(x)$ contains exactly n cycles, n is odd.

$\Rightarrow$ $\phi(x)$ is an odd permutation.

As $\phi(G) \le S_{2n}$ and $\phi(G)$ has an odd permutation

Then half of $\phi(G)$ are even and half are odd.

Now A be the set of even permutation in $\phi(G)$

then $|\phi(G) : A| = 2$ (Corollary 5 of sec 4.2).

$\Rightarrow$ A is normal

and also, $\phi(x) \notin A$

$\Rightarrow$ $\phi(G)$ is not simple.

$\Rightarrow$ G is not simple.

Example:- Let $|G| = 210 = 2 \times 105$

$\Rightarrow$ G is not simple.

Generalized Cayley's theorem:-

Let G be a group and H a subgroup of G. Let S be the group of all permutations of the left cosets of H in G. Then there is a homomorphism from G into S whose kernel lies in H and contains every normal subgp of G that is contained in H.

Proof: Same as theorem 3 of sec 4.2. (Dummit)

Here $S = \Pi_H(G)$ $\qquad G \times A \to A$

$A \to$ set of all left cosets of H.

2

Corollary 1 Index theorem!

Let G is a finite group and H is a proper subgp of G such that $|G|$ does not divide $|G:H|!$, then H contains a non-trivial normal subgp of G. In particular G is not simple.

Proof!- Let $|G:H| = n$

$|G|$ does not divide $n!$.

Let A = set of all left cosets of H in G.

G acts on A by left multiplication.

and π_H be the associated permutation representation

$$\pi_H : G \to S_n \quad \text{(homomorphism.)}$$

$\ker \pi_H \subseteq H$

Claim!- $\ker \pi_H \neq \{e\}$.

Let $\ker \pi_H = \{e\}$.

then $G \approx \phi(\pi_H) \leq S_n$.

$\Rightarrow$ $|G| = |\phi(\pi_H)|$

and as $\phi(\pi_H)$ is subgp of S_n.

$\Rightarrow$ $|\phi(\pi_H)| \,/\, n!$

$\Rightarrow$ $|G| \,/\, n!$ $\longrightarrow$ Contradiction.

$\Rightarrow$ $\ker \pi_H \neq \{e\}$ also $\ker \pi_H \neq G$.

and $\ker \pi_H$ is normal subgp of G.

$\Rightarrow$ G is not simple.

Ex:- Let $|G| = 216 = 2^3 3^3$

Then G has subgp of order 27 say H.

then $|G:H| = \frac{|G|}{|H|} = 8$.

and $|G|$ does not divide $|G:H|!$

i.e. 216 does not divies $8!$.

$\Rightarrow$ G is not simple. (By Corollary 1)

Corollary 2:- Embedding theorem:-

If a finite non-abelian simple group G has a subgroup of index n, then G is isomorphic to a subgroup of A_n.

Proof:- Let H be a subgroup of index n.

Claim:- $G \simeq$ subgp of A_n.

Let A be the set of all left cosets of H in G.

and G acts on A by left multiplication

and π_H be the associated permutation representation

$\pi_H : G \longrightarrow S_n$. homomorphism.

As G is simple, $\ker \pi_H = \{e\}$.

$\Rightarrow$ ~~$\phi(G)$ is~~ $\pi_H(G) = G'$ is a subgp of S_n.

$\Rightarrow$ G' has either even permutation or half even and half odd permutations.

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Now, if G' has half even and half odd permutation then the set of even permutation is normal in G' $\longrightarrow$ Contradiction.

Hence, G' has all even permutation.

Therefore $G' \leq A_n$.

as $G \cong G' \leq A_n$

$\rightarrow$ G is isomorphic to subgp of A_n.

Propⁿ 21:- If $|G| = 60$ and G has more than one sylow 5- subgp, then G is simple.

Proof:- Suppose by way of contradiction that $|G| = 60$ and $n_5 > 1$, then $n_5 = 6$.

Let $P \in syl_5(G)$ and H be a non-trivial normal subgroup of G. then $|H| / |G|$. $\Rightarrow$ $|H| / 60$.

If $5 / |H|$, then H contains sylow-5-subgroup of G syQ and since H is normal.

$$P = xQx^{-1} \leq H \quad \text{for some } x \in G.$$

$\Rightarrow$ H contains all 6 conjugates of this group.

Hence, $|H| \geq 1 + 6 \cdot 4 = 25$.

$\Rightarrow$ $|H| = 30$, but this is contradiction

As every gp of order 30 has only one sylow 5-subgp but H has 6 sylow 5-subgp.

$\rightarrow$ 5 does not divide $|H|$.

If $|H| = 6$ or 12, then H has a normal, characteristic sylow subgp say H' which is therefore normal in G. Replace H by H'. Then $|H'| = 2, 3$ or 4.

Let $\bar{G} = \frac{G}{H}$ so $|\bar{G}| = 30, 20$ or 15 respectively.

In each cases, $\bar{G}$ has a normal sylow 5-subgp say $\bar{P}$.

Let H_1 be the complete preimage of $\bar{P}$ in G then $H_1 \trianglelefteq G$ and $H_1 \neq G$ and $5 \mid |H_1|$ which is contradiction. (contradicts above para)

$\Rightarrow$ H is trivial.

Hence, G is simple.

Corollary:- A_5 is simple.

Proof:- $\langle(12345)\rangle$ and $\langle(13245)\rangle$ are distinct sylow 5-subgps of A_5. Hence A_5 is simple.

$\langle(12345)\rangle = \{(1), (12345), (13524), (14253), (15432)\}$

$\langle(13245)\rangle = \{(1), (13245), (12534), (14352), (15423)\}$

Thm:- A_n is simple for all $n \geq 5$.