Thm 2.1 Let V and W be vector spaces and $T: V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of V and W respectively.

Proof:- Let $T: V \to W$ be linear.

Claim:- $N(T)$ is a subspace of V.

(i) By property ①, As T is linear,

$$T(0) = 0$$

[for ~~sym~~ notations let 0_V and 0_W are additive identity of V and W respectively.]

$$\therefore \quad T(0_V) = 0_W$$

$$\Rightarrow \quad 0_V \in N(T).$$

(ii) Let $x, y \in N(T)$

$$\Rightarrow T(x) = T(y) = 0_W$$

then $T(x+y) = T(x) + T(y)$ $(\because T \text{ is linear})$

$$= 0_W + 0_W$$

$$= 0_W$$

$$\Rightarrow \quad x+y \in N(T)$$

(iii) Let $c \in F$ and $x \in N(T) \Rightarrow T(x) = 0_W$.

$$T(cx) = cT(x)$$

$$= c.0_W$$

$$= 0_W$$

$$\Rightarrow \quad cx \in N(T)$$

$\therefore$ $N(T)$ is a subspace of V.

Claim!- $R(T)$ is a subspace of W.

(i) As $T(0_V) = 0_W$

$\Rightarrow \quad 0_W \in R(T)$

(ii) Let $x, y \in R(T)$, then there exists u and w in V such that

$T(u) = x \quad$ and $\quad T(v) = y$.

then $T(u+v) = T(u) + T(v) \quad (\because T \text{ is linear})$

$= x + y$

$\Rightarrow \quad x + y \in R(T)$.

(iii) Let $c \in F$ and $x \in R(T) \Rightarrow \exists \ u \in V$ s.t. $T(u) = x$.

then $T(cu) = cT(u)$

$= cx$

$\therefore \quad T(cu) = cx$

$\Rightarrow \quad cx \in R(T)$, whenever $c \in F$ & $x \in R(T)$

Thus $R(T)$ is a subspace of W. ■

Thm 2.2.!- Let V and W be vector spaces, and let $T: V \to W$ be linear. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then

$$R(T) = \text{Span}(T(\beta)) = \text{Span}(\{T(v_1), T(v_2), \ldots, T(v_n)\}).$$

Proof!- As $v_i \in V \quad \forall \ i = 1, 2, \ldots, n$

$\Rightarrow \quad T(v_i) \in R(T) \quad \forall \ i = 1, 2, \ldots, n$

$\Rightarrow \quad \{T(v_1), T(v_2), \ldots, T(v_n)\} \subseteq R(T)$

$\Rightarrow \quad \text{Span}(\{T(v_1), \ldots, T(v_n)\}) \subseteq R(T) \quad$ (By thm 1.5)

$\therefore$ $Span(T(\beta)) \subseteq R(T)$ —(1)

Now let $w \in R(T)$, then there exists $v \in V$ such that $T(v) = w$

As $v \in V$

$$\therefore \quad v = \sum_{i=1}^{n} a_i v_i \quad \text{for some } a_i \in F, \ i = 1, 2, \to n.$$

$$\Rightarrow T(v) = T\left(\sum_{i=1}^{n} a_i v_i\right)$$

$$= \sum_{i=1}^{n} a_i T(v_i)$$

$$\Rightarrow \quad w = T(v) = \sum_{i=1}^{n} a_i T(v_i)$$

$\therefore$ w is a linear combination of vectors $\{T(v_1), \text{—}, T(v_n)\}$

$\Rightarrow$ $w \in Span(\{T(v_1), \to T(v_n)\})$

As $w \in R(T)$ is arb.

$\Rightarrow$ $R(T) \subseteq span(\{T(v_1), \to T(v_n)\}) = span(T(\beta))$ —(2)

from eq. (1) and eq. (2), we have

$$R(T) = span(\{T(v_1), \to T(v_n)\}) = Span(T(\beta)). \blacksquare$$

Def^n:- Let V and W be vector spaces, and $T: V \to W$ be linear, If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of T, denoted by nullity(T), to be dimension of $N(T)$ and rank of T, denoted by rank(T), to be dimension of $R(T)$.

Examples:-

(1) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$$

then T is linear and

$$N(T) = \{(0,0)\}$$

and $R(T) = \{(x, 0, y) : (\frac{x+y}{3}, \frac{2x-y}{3}) \in \mathbb{R}^2\}$.

Therefore, nullity$(T) = 0 = \dim(N(T))$.

and Rank$(T) = \dim(R(T)) = 2$.

(2) Let $T: P_2(\mathbb{R}) \longrightarrow M_{2\times 2}(\mathbb{R})$ defined by

$$T(f(x)) = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

then T is linear (show yourself).

$N(T) = \{f(x) \in P_2(\mathbb{R}) : T(f(x)) = 0\}$.

Let $f(x) \in P_2(\mathbb{R})$ s.t. $T(f(x)) = 0$

then $f(x) = a_0 + a_1 x + a_2 x^2$ — (1)

As $T(f(x)) = 0$

$$\Rightarrow \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\Rightarrow f(1) = f(2)$ and $f(0) = 0$

Putting values in eq. (1), we get

$a_0 = 0$ and $a_1 = -3a_2$

$\therefore f(x) = -3a_2 x + a_2 x^2$

Thus $N(T) = \{ f(x) \in P_2(\mathbb{R}) : f(x) = -3a_2 x + a_2 x^2 \}$

$= \text{span}\{ -3x + x^2 \}$

$\Rightarrow \dim(N(T)) = 1$

$\Rightarrow \text{Nullity}(T) = 1$

Since $\beta = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$

$\therefore R(T) = \text{span}(T(\beta))$

$= \text{span}(\{T(1), T(x), T(x^2)\})$

$$= \text{span}\left(\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}}_{L.D.} \right\}\right)$$

$$= \text{span}\left(\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \right\}\right).$$

$\therefore \text{Rank}(T) = \dim(R(T)) = 2$.

Thm 2.3 (Dimension theorem) :- Let V and W be vector spaces and let $T: V \to W$ be linear. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

Proof :- As V is finite-dimensional and $N(T)$ is a subspace of V, Therefore $N(T)$ is also finite dimensional.

Let $\dim(V) = n$ and $\dim(N(T)) = \text{nullity}(T) = k$.

Let $\{v_1, v_2, \ldots, v_k\}$ be the basis of $N(T)$, then we may extend ~~it~~ it to a basis $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V [By Theorem 1.11 Corollary].

Let $S = \{T(v_{k+1}), \ldots, T(v_n)\}$

Claim:- S is a basis for $R(T)$.

first we show that S generates $R(T)$

i.e. $R(T) = \text{span}(S)$..

As $\beta = \{v_1, v_2, \ldots, v_n\}$ is basis for V.

$\therefore R(T) = \text{span}(T(\beta))$ [By thm 2.2)

$\Rightarrow R(T) = \text{span}(\{T(v_1), \ldots, T(v_k), T(v_{k+1}), \ldots, T(v_n)\})$

$= \text{span}(\{0, \ldots, 0, T(v_{k+1}), \ldots, T(v_n)\})$

$\left(\begin{array}{l} \because v_i \in N(T), \ i=1,2,\ldots,k \\ \Rightarrow T(v_i) = 0, \ i=1,2,\ldots,k \end{array} \right)$

$\Rightarrow R(T) = \text{span}(\{T(v_{k+1}), \ldots, T(v_n)\})$

$R(T) = \text{span}(S)$

$\Rightarrow$ S generates $R(T)$.

Now we show that S is linearly independent.

Let $a_{k+1}, a_{k+2}, \ldots, a_n \in F$ such that

$a_{k+1} T(v_{k+1}) + \ldots + a_n T(v_n) = 0$

$\Rightarrow \sum_{i=k+1}^{n} a_i T(v_i) = 0$

$$\Rightarrow \sum_{i=k+1}^{n} T(a_i v_i) = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} a_i v_i\right) = 0$$

$$\Rightarrow \sum_{i=k+1}^{n} a_i v_i \in N(T) = \text{span}(\{v_1, v_2, \dots, v_k\})$$

$\therefore$ There exists $b_1, b_2, \dots, b_k \in F$ such that

$$\sum_{i=k+1}^{n} a_i v_i = \sum_{i=1}^{k} b_i v_i$$

$$\Rightarrow \sum_{i=k+1}^{n} a_i v_i - \sum_{i=1}^{k} b_i v_i = 0$$

$$\Rightarrow (-b_1)v_1 + (-b_2)v_2 + \dots + (-b_k)v_k + a_{k+1}v_{k+1} + \dots + a_n v_n = 0$$

As $\{v_1, v_2, \dots, v_n\}$ is basis of V and hence linearly independent.

Therefore all b_i's and a_i's are zero.

$$\Rightarrow a_i = 0 \quad \forall\ i = k+1, \dots, n.$$

$\therefore$ S is linearly independent.

$\Rightarrow$ S is a basis of $R(T)$

$\therefore$ $\text{Rank}(T) = n-k$. ($\because$ As S have $n-k$ vectors)

which proves the theorem. ∎

<u>Q</u>. Verify dimension theorem for examples we have done.

Thm 2.4 Let V and W be vector spaces, and let $T: V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof:- Let $T: V \to W$ is one-to-one.

Claim:- $N(T) = \{0\}$.

Let $x \in N(T)$

$\Rightarrow \quad T(x) = 0$

Also, $\quad T(0) = 0 \qquad (\because T \text{ is linear})$

$\Rightarrow \quad T(x) = T(0)$

$\Rightarrow \quad x = 0 \qquad (\because T \text{ is } 1\text{-}1)$.

$\therefore \quad N(T) = \{0\}$

Conversely, let $N(T) = \{0\}$

Claim:- T is one-to-one.

Let $x, y \in V$ such that $T(x) = T(y)$

$\Rightarrow \quad T(x) - T(y) = 0$

$\Rightarrow \quad T(x-y) = 0 \qquad (\because T \text{ is linear})$

$\Rightarrow \quad x-y \in N(T)$

$\Rightarrow \quad x-y = 0 \qquad (\because N(T) = \{0\})$

$\Rightarrow \quad x = y$.

$\therefore$ T is one-to-one. ■

Thm 2.5 :- Let V and W be vector spaces of equal (finite) dimension, and let $T: V \to W$ be linear. Then the following are equivalent,

a) T is one-to-one

b) T is onto

c) $\text{rank}(T) = \dim(V)$.

Proof :- (We will prove the theorem by showing (a) $\Leftrightarrow$ (c) and (b) $\Leftrightarrow$ (c).)

Let T is one-to-one

$\Leftrightarrow \quad N(T) = \{0\}$ $\quad$ ($\because$ By thm 2.4).

$\Leftrightarrow \quad \text{nullity}(T) = 0$

$\Leftrightarrow \quad \text{rank}(T) = \dim(V)$ $\quad$ ($\because$ By dimension thm)

Therefore we have (a) $\Leftrightarrow$ (c).

Now let T is onto.

$\Leftrightarrow \quad R(T) = W$

$\Leftrightarrow \quad \text{rank}(T) = \dim(W) = \dim(V)$ $\quad$ ($\because \dim(V) = \dim(W)$)

$\Leftrightarrow \quad \text{rank}(T) = \dim(V)$

$\therefore$ we have (b) $\Leftrightarrow$ (c) ∎.

Note :- In theorem 2.5, the dimension of V and W are equal and finite. If they are not equal or finite then the result may not hold.

Example.

(1) Let $T: P(\mathbb{R}) \to P(\mathbb{R})$ defined by

$$T(f(x)) = f'(x)$$

then T is linear (show it).

Here $P(\mathbb{R})$ is ring of polynomials of real coefficients.

Prove that T is onto, but not one-to-one.

Soln:- As $f(x) = 2$ and $g(x) = 3$ are in $P(\mathbb{R})$

and $T(f(x)) = T(g(x)) = 0$

but $f(x) \neq g(x)$

$\therefore$ T is not one-to-one.

Claim:- T is onto.

Let $g(x) \in P(\mathbb{R})$, then $g(x)$ is continuous real valued function.

$\Rightarrow$ $g(x)$ is integrable and

$\int_0^x g(t)\,dt$ is again an polynomial

and let $f(x) = \int_0^x g(t)\,dt$

then $T(f(x)) = g(x)$

$\Rightarrow$ T is onto.

Note:- Thm 2.5 does not hold, as $P(\mathbb{R})$ is not finite dimensional.

(2) Let $T: P_2(R) \to R^3$ defined by

$$T(a_0 + a_1 x + a_2 x^2) = (a_0, a_1, a_2)$$

then T is linear (Do it)

and T is one-to-one (Do it)

$\therefore$ By thm 2.5 (As $P_2(R)$ and R^3 are finite and have equal dimension)

T is onto

and $\text{rank}(T) = \dim(P_2(R)) = 3$.

Thm 2.6 :- Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ is basis for V. For $w_1, w_2, \ldots, w_n$ in W, there exists exactly one linear transformation $T: V \to W$ such that
$T(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

Proof :- Let $x \in V$ and as $\{v_1, v_2, \ldots, v_n\}$ is basis for V, x can be uniquely written as lineer combination of vectors $\{v_1, v_2, \ldots, v_n\}$

$\Rightarrow$ there exists unique scalars $a_1, a_2, \ldots, a_n \in F$ s.t.

$$x = \sum_{i=1}^{n} a_i v_i \quad -(1)$$

Define $T: V \to W$ as

$$T(x) := \sum_{i=1}^{n} a_i w_i$$

where a_i's are from eq. (1),

Then T is well-defined (Show it)

Now we show that T is required transformation for that we have to show that T is linear, $T(v_i) = w_i \ \forall i$ and T is unique.

(i) <u>T is linear</u>

Let $x, y \in V$ and $c \in F$

then $x = \sum_{i=1}^{n} a_i v_i$ and $y = \sum_{i=1}^{n} b_i v_i$

and $cx + y = \sum_{i=1}^{n} c a_i v_i + \sum_{i=1}^{n} b_i v_i$

$= \sum_{i=1}^{n} (c a_i + b_i) v_i$

$\therefore T(cx+y) = \sum_{i=1}^{n} (c a_i + b_i) w_i$

$= c \sum_{i=1}^{n} a_i w_i + \sum_{i=1}^{n} b_i w_i$

$= c\,T(x) + T(y)$

$\Rightarrow T$ is linear.

(ii) <u>$T(v_i) = w_i$</u> $\forall i$

As $v_i = 1 \cdot v_i \quad \forall i = 1, 2, \ldots, n$

$\therefore T(v_i) = 1 \cdot w_i = w_i$

$\Rightarrow T(v_i) = w_i \quad \forall i = 1, 2, \ldots, n.$

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(iii) T is unique :-

Let $U : V \to W$ be linear such that $U(v_i) = w_i \; \forall i$.

Claim :- $U = T$

Let $x \in V$, then $x = \sum_{i=1}^{n} a_i v_i$

for scalars $a_1, a_2, \ldots, a_n$.

then $U(x) = U\left(\sum_{i=1}^{n} a_i v_i\right)$

$= \sum_{i=1}^{n} a_i U(v_i)$ $\quad$ [$\because$ U is linear)

$= \sum_{i=1}^{n} a_i w_i$ $\quad$ [$\because$ $U(v_i) = w_i$)

$= T(x)$

$\Rightarrow$ $U(x) = T(x)$

As $x \in V$ is arb.

$\therefore$ $U = T$ ■

Corollary :- Let V and W be vector spaces, and suppose that V has a finite basis $\{v_1, v_2, \ldots, v_n\}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \ldots, n$, then $U = T$.

Proof :- Let $w_i = T(v_i)$, $i = 1, 2, \ldots, 3$

Then By theorem 2.6

$U = T$ ■