Partial Differential Equations :-

A differential equation involving partial derivatives w.r.to more than one ~~varia~~ independent variables.

E.g. (1) $\dfrac{\partial^2 v}{\partial t^2} = k\left(\dfrac{\partial^3 v}{\partial x^3}\right)^2$ where $v = v(x,t)$

(2) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$, $u = u(x,y,z)$.

(3) $x\dfrac{\partial z}{\partial x} = y\dfrac{\partial z}{\partial y}$, where $z = z(x,y)$.

Note:- The order of the highest order derivative involve in P.D.E. is order of P.D.E. and power of the highest order derivative is called degree of P.D.E.

Linear Partial D.E. :- A P.D.E. is said to be linear if, it is of first degree in the dependent variable and its partial derivatives (i.e. powers or products of dependent variables and/or its partial derivatives must be absent.)

Note:- When $z = z(x,y)$, where x & y are independent variables & z is dependent.

then we denote

$$\frac{\partial z}{\partial x} = p,\quad \frac{\partial z}{\partial y} = q,\quad \frac{\partial^2 z}{\partial x^2} = r,\quad \frac{\partial^2 z}{\partial x \partial y} = s$$

& $\dfrac{\partial^2 z}{\partial y^2} = t$.

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Formation of first order PDE :-

Rule I :- Formation of a PDE by the elimination of arbitrary constants

Consider an equation

$$f(x, y, z, a, b) = 0 \quad \text{——— (I)}$$

where a and b are arbitrary constants. Let z be regarded as function of two independent variables x and y.

Differentiating (I) w.r.to x and y partially we get

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z} = 0 \quad \text{——— (2)}$$

Eliminating a and b from (I) & (2) we get an equation of the form

$$f(x, y, z, p, q) = 0 \quad \text{——— (3)}$$

~~Showing that~~ which is a P.D.E. of first order.

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Example :- (1) Consider the equation

$$x^2+y^2+(z-c)^2 = a^2 \quad \text{--- (1)}$$

where a & c are arbitrary constants.

Differentiating (1) w.r.to x and y, we get

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad x + p(z-c) = 0 \quad \text{--- (2)}$$

and

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad y + q(z-c) = 0 \quad \text{--- (3)}$$

Now, eliminating 'c' between (2) & (3) we get

$$\boxed{yp - xq = 0} \quad \text{--- (4)}$$

Note :- Eqn (1) represents the set of all spheres whose centers lies along the z-axis. Then eqn (4) is P.D.E. of the set of all spheres whose centers lie along the z-axis.

Example (2) :- ~~x^2+y^2~~

Exercise. Find a P.D.E. for the surface

$$x^2+y^2=(z-c)^2 \tan^2\alpha$$

where c and α are arbitrary constants.

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(5) Exercise (2) Find P.D.E of following surface

$$z = f(x^2+y^2) \quad \text{—— (I)}$$

where f is arbitrary function.

Differentiate eqⁿ (1) with r.t.o x & y

$$\frac{\partial z}{\partial x} = 2x f'(x^2+y^2) \quad \text{or} \quad p = 2x f'(x^2+y^2)$$

and

$$\frac{\partial z}{\partial y} = 2y f'(x^2+y^2) \quad \text{or} \quad q = 2y f'(x^2+y^2)$$

then

$$\frac{p}{q} = \frac{2x f'(x^2+y^2)}{2y f'(x^2+y^2)}$$

or

$$\frac{p}{q} = \frac{x}{y}$$

or

$$\boxed{yp - qx = 0}$$

which is required P.D.E. of first order.

Exercise (3) Eliminate the constants a & b from the following equations

(a) $z = (x+a)(y+b)$ (b) $2z = (ax+y)^2 + b$

(c) $ax^2 + by^2 + z^2 = 1$

(4) Eliminate the arbitrary function f from the eqⁿs:

(a) $z = xy + f(x^2+y^2)$ (b) $z = x+y+f(xy)$

(c) $z = f\left(\frac{xy}{z}\right)$ (d) $z = f(x, y)$

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Rule II:- Formations of PDE by eliminating of arbitrary function F ~~from~~ the equation $F(u, v) = 0$ — (1)

where u & v are functions of x, y & z.

Differentiating (1) w.r.to x & y, partially, we get

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x} + p\frac{\partial F}{\partial z}\right\} + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \quad \text{— (2)}$$

and

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial F}{\partial z}\right) + \frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \quad \text{— (3)}$$

from (2)

$$\frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v} = -\frac{\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)} \quad \text{— (4)}$$

and from (4)

$$\frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v} = -\frac{\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)} \quad \text{— (5)}$$

from (4) & (5), we obtain

$$-\frac{\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)} = -\frac{\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)}{\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)}$$

so

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)$$

which is required PDE of First order.

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Example :- $F(x+y+z,\ x^2+y^2-z^2)=0$ — (1)

Let $u = x+y+z$, $v = x^2+y^2-z^2$

then eqn (1) becomes $\frac{\partial u}{\partial x}=1,\ \frac{\partial u}{\partial y}=1,\ \frac{\partial u}{\partial z}=1$ & $\frac{\partial v}{\partial x}=2x,\ \frac{\partial v}{\partial y}=2y$

$$F(u,v)=0 \quad \text{— (2)} \qquad \frac{\partial v}{\partial z}=-2z.$$

Differentiating eqn (2) w.r.to x partially, we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

$$\frac{\partial F}{\partial u}(1+p)+\frac{\partial F}{\partial v}(2x-2zp)=0$$

then

$$\frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v}=\frac{-2(x-zp)}{1+p} \quad \text{— (3)}$$

Again, Differentiating eqn (2) with r. to y, partially we get

$$\frac{\partial F}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial F}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$$

$$\frac{\partial F}{\partial u}(1+q)+\frac{\partial F}{\partial v}(2y-2zq)=0$$

or

$$\frac{\partial F}{\partial u}\Big/\frac{\partial F}{\partial v}=\frac{-2(y-zq)}{1+q} \quad \text{— (4)}$$

from (3) & (4)

$$\frac{-2(x-zq)}{1+p}=\frac{-2(y-zq)}{1+q}$$

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$$(1+q)(x-zp) = (1+p)(y-qz)$$

or

$$x+qx-zp-qpz = y+py-qz-pqz$$

$$py+zp-xq-qz = x-y$$

$$\boxed{(y+z)p-(x+z)q = x-y}$$

which is required P.D.E.

Exercise :- Eliminate the arbitrary function f from the eqn

$$f(x^2+y^2+z^2,\ z^2-2xy) = 0$$

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