Complex Analysis

Sec 60. Laurent Series.

If a function f fails to be analytic at a point $z = z_0$ (singular point), one can't apply Taylor's Theorem at that point (i.e. f can't be expanded in a power series with z_0 as center).

However, it is possible to find series representation for $f(z)$ involving both positive and negative powers of $z - z_0$.

To understand the above point, let us consider the following example:

The function $f(z) = \frac{\sin z}{z^4}$ is not analytic at the isolated singularity $z = 0$ and hence can't be expanded in a Maclaurin series (Note that here $z_0 = 0$).

However, $\sin z$ is an entire function (analytic throughout $\mathbb{C}$) and we know that its Maclaurin series is given by

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

(see the module of Taylor's Series)

which converges for $|z| < \infty$.

Hence, when $0<|z|<\infty$

$$f(z)=\frac{\sin z}{z^4}=\frac{1}{z^4}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right)$$

$$=\frac{1}{z^3}-\frac{1}{3!z}+\frac{z}{5!}-\frac{z^3}{7!}+\cdots \qquad (*)$$

I again want to emphasize that the above series is not the Maclaurin series ~~of~~ for $f(z)$ but we are able to write $f(z)$ into a series involving powers of z. (or powers of $(z-z_0)$ where $z_0=0$).

We call such terms as $\frac{1}{z^3}$ and $\frac{1}{z}$ negative powers of z since they can be written z^{-3} and z^{-1}, respectively.

Let us again write $(*)$,

$$f(z)=\frac{\sin z}{z^4}=\underbrace{\frac{1}{z^3}-\frac{1}{6z}}_{\text{principal part}}+\underbrace{\frac{z}{5!}-\frac{z^3}{7!}+\cdots}_{\text{analytic part}} \qquad (**)$$

The part consisting of non-negative powers of $(z-0)$ is called the analytic part ~~of~~ ~~the series~~ of the series $(**)$ and will converge for $|z-0|<\infty$ and the part consisting ~~of~~ with negative powers of $(z-0)$ is called the principal part of the series $(**)$ and will ~~converge~~ be valid for $|z|>0$.

Thus $(**)$ converges for all z except at $z=0$; i.e. the series representation is valid for $0<|z|<\infty$.

The similar examples were also discussed in the module on Taylor Series. (Please see Example 5 (Page 10), Q11, Q12, Q13 (Pages 17-18)).

Theorem. Suppose that a function f is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any +vely oriented simple closed contour around z_0 and lying in that domain. Then at each point in the domain, $f(z)$ has the series representation

(1) $$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad (R_1 < |z-z_0| < R_2)$$

where

(2) $$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n=0,1,2,\ldots)$$

and

(3) $$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}} \quad (n=1,2,3,\ldots)$$

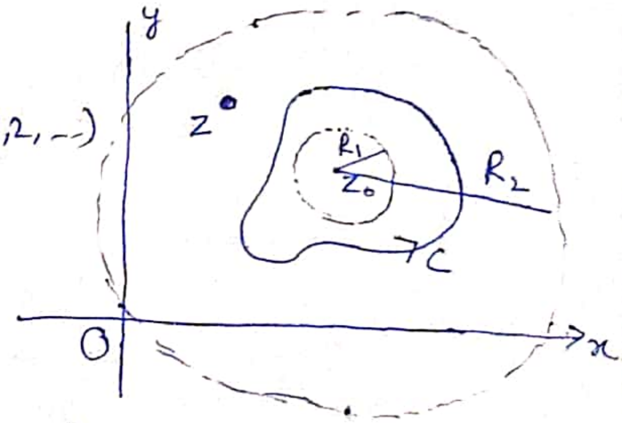


Fig. 1.

The series (1) can be re-arranged as

(4) $$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad (R_1 < |z-z_0| < R_2)$$

where $$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n=0,\pm1,\pm2,\ldots)$$

In either one of the forms (1) and (4), the representation of $f(z)$ is called <u>Laurent Series</u>.

Details regarding how to obtain (4) from (1) are as follows:

Replace 'n' by '$-n$' in the second series in representation (1) enables us to write that series as

$$\sum_{n=-\infty}^{-1} \frac{b_{-n}}{(z-z_0)^{-n}}$$

where

$$b_{-n} = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \qquad (n=-1,-2,-3,\dots)$$

Thus

$$f(z) = \sum_{n=-\infty}^{-1} b_{-n}(z-z_0)^n + \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{—}(*)$$

$$(R_1 < |z-z_0| < R_2)$$

If

$$c_n = \begin{cases} b_{-n} & \text{when } n \le -1 \\ a_n & \text{when } n \ge 0 \end{cases}$$

$(*)$ becomes

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \qquad (R_1 < |z-z_0| < R_2)$$

where $c_n = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$ $(n=0,\pm1,\pm2,\dots)$

Scanned with CamScanner

Remark 1. We can write (4) as the sum of two series

$$f(z) = \sum_{k=1}^{\infty} c_k (z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k \quad \text{— (A)}$$

The two series on the R.H.S. in (A) are given special names in the literature. The part with negative powers of $(z-z_0)$ i.e.

$$\sum_{k=1}^{\infty} c_{-k} (z-z_0)^{-k} = \sum_{k=1}^{\infty} \frac{c_k}{(z-z_0)^k} \quad \text{— (B)}$$

is called the principal part of the series (4) and will converge for $\left|\frac{1}{(z-z_0)}\right| < \frac{1}{R_1}$ or equivalenty for $|(z-z_0)| > R_1$.

The part consisting of the nonnegative powers of $z-z_0$,

$$\sum_{k=0}^{\infty} c_k (z-z_0)^k \quad \text{— (C)}$$

is called the analytic part of the series (4) and will converge for $|z-z_0| < R_2$.

Hence the sum of (B) and (C) converges when z satisfies both $|z-z_0| > R_1$ and $|z-z_0| < R_2$ i.e. when z is ~~satisfies~~ a point in an annular domain defined by $R_1 < |z-z_0| < R_2$.

Remark 2. The annular domain in the theorem defined by $R_1 < |z-z_0| < R_2$ need not have "ring" shape as illustrated in the Fig 1. Here are some other possible annular domains:

(a) $R_1 = 0$, R_2 finite $= R$ (say) (b) $R_1 \neq 0$ $R_2 = \infty$ (c) $R_1 = 0$, $R_2 = \infty$

In the case (a), the series converges in annular domain defined by $0 < |z-z_0| < R$.

This is the interior of the circle $|z-z_0| = R$ except the point $z = z_0$.

In other words, the domain is a punctured open disk.

In the case (b), the annular domain is defined by $R_1 < |z-z_0|$ and consists of all points exterior to the circle $|z-z_0| = R_1$.

In the case (c), domain is defined by $0 < |z-z_0|$.

This represents entire complex plane except the point z_0.

Remark 3. The integral formula in (4) for the coefficients of a Laurent series is rarely used in actual practice. In many instances, we can obtain a desired Laurent series either by apply a known power series expansion of function or by making use of geometric series..

$\left(\frac{1}{1-z} = 1 + z + z^2 + \cdots \cdots \right.$, when $\left. |z| < 1 \right)$

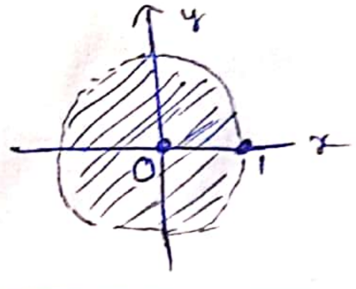
Examples

Example 1. Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains:

(a) $0<|z|<1$ (b) $1<|z|$ (c) $0<|z-1|<1$ (d) $1<|z-1|$.

The given function $f(z)$ has two isolated singularities $z=0$ and $z=1$.

(a) $0<|z|<1$



Let us write

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} \cdot \frac{1}{(1-z)}$$

We know that

$$\frac{1}{1-z} = 1+z+z^2+\cdots \qquad \text{when } |z|<1$$

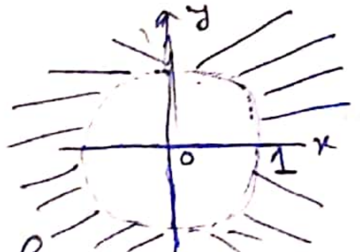
Now, when $0<|z|<1$

$$f(z) = -\frac{1}{z}\left(1+z+z^2+\cdots\right)$$

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots \qquad \text{--- } (*)$$

$(*)$ is Laurent series for $f(z)$ which converges for $0<|z|<1$.

(b) $1<|z|$.



To obtain a series that converges for $1<|z|$, we start by constructing series that converges for $|1/z|<1$.

We can write the given function f as

$$f(z) = \frac{1}{z^2} \cdot \frac{1}{\left(1-\frac{1}{z}\right)}$$

Now

$$\frac{1}{1-\frac{1}{z}} = 1+\frac{1}{z}+\frac{1}{z^2}+\cdots \quad \text{when } \left|\frac{1}{z}\right|<1$$

i.e. when $|z|>1$.

Thus the required Laurent series is

$$f(z) = \frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

$$f(z) = \frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots \quad \text{when } |z|>1.$$

Note that series for $f(z)$ consists of only principal part (i.e. expansion of $f(z)$ consists of negative powers of z).

(c). $0<|z-1|<1$.

If we compare this case with (a), we note that in this case we want to expand $f(z)$ in the powers of $(z-1)$.

By adding and subtracting '1' in the denominator, $f(z)$ can be written as

$$f(z) = \frac{1}{(1-1+z)(z-1)}$$

$$= \frac{1}{(z-1)}\cdot\frac{1}{1+(z-1)}$$

Now, $\dfrac{1}{1+(z-1)} = \left[1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right]$

when $|z-1|<1$.

Hence, when $0 < |z-1| < 1$

$$f(z) = \frac{1}{(z-1)}\left[1-(z-1)+(z-1)^2+\cdots\right]$$

$$= \frac{1}{(z-1)} - 1 + (z-1) - (z-1)^2 + \cdots$$

(d) $1 < |z-1|$

This part is similar to part (b).

We write 'f' as

$$f(z) = \frac{1}{(z-1)} \cdot \frac{1}{1+(z-1)}$$

$$= \frac{1}{(z-1)^2}\left[\frac{1}{1+\frac{1}{(z-1)}}\right]$$

Now $\dfrac{1}{1+\frac{1}{z-1}} = 1 - \dfrac{1}{z-1} + \dfrac{1}{(z-1)^2} - \dfrac{1}{(z-1)^3} + \cdots$

when $\left|\dfrac{1}{z-1}\right| < 1$

i.e. $|z-1| > 1$.

$\therefore$ for $|z-1| > 1$

$$f(z) = \frac{1}{(z-1)^2}\left[1 - \frac{1}{(z-1)} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \cdots\right]$$

$$f(z) = \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \frac{1}{(z-1)^5} + \cdots$$

Example 2. Expand $f(z) = \dfrac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for (a) $0 < |z-1| < 2$.

(b) $0 < |z-3| < 2$.

(a) $0 < |z-1| < 2$ Here $z_0 = 1$. $R_1 = 0, R_2 = 2$.

Now, $f(z) = \dfrac{1}{(z-1)^2(z-3)} = \dfrac{1}{(z-1)^2(-2+(z-1))}$

$$= \frac{-1}{2(z-1)^2} \frac{1}{1-\left(\frac{z-1}{2}\right)}$$

Now For $\left|\frac{z-1}{2}\right| < 1$ i.e. when $|z-1| < 2$.

$$\frac{1}{1-\left(\frac{z-1}{2}\right)} = 1 + \frac{z-1}{2} + \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} + \cdots$$

$\therefore$ For $0 < |z-1| < 2$

$$f(z) = \frac{-1}{2(z-1)^2}\left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{4} + \frac{(z-1)^3}{8} + \cdots\right]$$

$$f(z) = \frac{-1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$$

(b) $0 < |z-3| < 2$, Here $z_0 = 3$, $R_1 = 0$, $R_2 = 2$

To obtain powers of $z-3$, we write $z-1 = 2 + (z-3)$

$$\therefore f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(2+z-3)^2(z-3)} = \frac{1}{(z-3)}\left[2+(z-3)\right]^{-2}$$

Scanned with CamScanner

$$= \frac{1}{4(z-3)}\left[1+\frac{z-3}{2}\right]^{-2}$$

Now, for $\left|\frac{z-3}{2}\right|<1$ i.e. when $|z-3|<2$

$$\left[1+\frac{z-3}{2}\right]^{-2} = 1+\frac{(-2)}{1!}\left(\frac{z-3}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2+\cdots$$

(Using Binomial Thm.).

$\therefore$ For $0<|z-3|<2$

$$f(z) = \frac{1}{4(z-3)}-\frac{1}{4}+\frac{3}{16}(z-3)-\frac{1}{8}(z-3)^2+\cdots$$

Example 3. Expand $f(z)=\frac{8z+1}{z(1-z)}$ in a Laurent Series valid for $0<|z|<1$.

Soln. By partial fractions, we can rewrite f as

$$f(z)=\frac{8z+1}{z(1-z)} = \frac{1}{z}+\frac{9}{1-z}$$

Now for $|z|<1$

$$\frac{9}{1-z} = 9(1+z+z^2+\cdots)$$

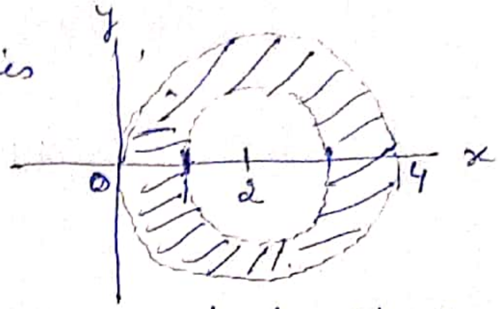
$\therefore$ For $0<|z|<1$

Laurent series for 'f' is given by

$$f(z)= \frac{1}{z}+9+9z+9z^2+\cdots.$$

Example 4. Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2| < 2$.

Soln. Our goal is to find two series involving integral powers of $z-2$, one converging for $1 < |z-2|$ and the other converging for $|z-2| < 2$.



$1 < |z-2| < 2$.

By using partial fraction, we can rewrite 'f' as

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z). \quad \text{———} (*)$$

Now, $f_1(z) = -\frac{1}{z} = -\frac{1}{2+z-2}$

$$= -\frac{1}{2} \frac{1}{1+\frac{z-2}{2}}$$

For $\left|\frac{z-2}{2}\right| < 1$ i.e. when $|z-2| < 2$

$$\frac{1}{1+\frac{z-2}{2}} = 1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{8} + \cdots$$

$$\therefore f_1(z) = -\frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \frac{(z-2)^3}{8} + \cdots\right]$$

when $|z-2| < 2$

i.e. $f_1(z) = -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$

when $|z-2| < 2$.

——— $(**)$

Now, $f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2} \cdot \frac{1}{1+\frac{1}{z-2}}$

For $\left|\frac{1}{z-2}\right| < 1$ i.e. when $|z-2| > 1$

$$\frac{1}{1+\frac{1}{z-2}} = 1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots$$

$\therefore$ For $|z-2| > 1$

$$f_2(z) = \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right]$$

$$f_2(z) = \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots$$

$\qquad\qquad$ —— (***)

Using (**) and (***) in (*), we get

$$f(z) = \cdots - \frac{1}{(z-2)^4} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)} - \frac{1}{2} + \frac{(z-2)}{2^2} - \frac{(z-2)^2}{2^3} + \cdots$$

~~for~~ when $1 < |z-2| < 2$

Example
Page 203-205
Brown & Churchill

Expand $f(z) = \frac{-1}{(z-1)(z-2)}$ in a Laurent Series valid for the following annular domains:

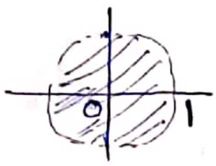
(a) $|z|<1$ (b) $1<|z|<2$ (c) $2<|z|<\infty$

Soln. We note that $f(z)$ has two singular points viz. $z=1$ and $z=2$.

$\therefore$ f is analytic in each of the domains mentioned in the parts (a), (b) & (c).

In each of the domains $f(z)$ has series representation in powers of z.

(a) when $|z|<1$



We can write

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{-1}{(1-z)} + \frac{1}{2(1-z/2)} \quad \text{—(A)}$$

It is given that $|z|<1$, which further implies that $|z|<2$ i.e. $|z/2|<1$.

$$\therefore \frac{1}{1-z} = 1+z+z^2+ \cdots \quad \text{when } |z|<1$$

$$\text{and} \quad \frac{1}{1-z/2} = 1+z/2+\frac{z^2}{2^2}+ \cdots \quad \text{when } |z/2|<1$$

—(X)

Using (*) in (A), we get

$$f(z) = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty} \frac{z^n}{2^n} \quad \text{when } |z|<1. \qquad \text{— (B)}$$

Re-arranging further, we can write (B) as

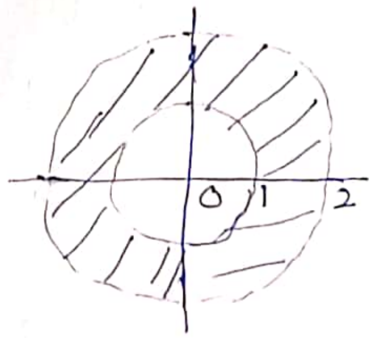
$$f(z) = -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1}-1)z^n$$

when $|z|<1$

— (**)

(**) is Laurent series for $f(z)$ which converges for $|z|<1$.

(b) $1<|z|<2$



Since $1<|z|<2$, we have

$\left|\frac{1}{z}\right|<1$ and $|z/2|<1$

Keeping above in mind, we can write f as

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2(1-z/2)} \qquad \text{— (C)}$$

Now,

$$\frac{1}{1-\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots, \text{ when } \left|\frac{1}{z}\right|<1 \text{ or } |z|>1$$

and

$$\frac{1}{1-z/2} = 1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots, \text{ when } |z/2|<1 \text{ or } |z|<2.$$

Using above in (C), we get

$$f(z) = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) + \frac{1}{2}\left(1+z/2+\frac{z^2}{2^2}+\cdots\right)$$

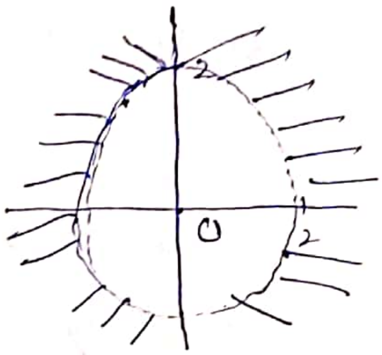
when $1<|z|<2$.

(16)

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad \text{when } (1<|z|<2) \qquad \text{—— } (\times\times\times)$$

$(\times\times\times)$ is a Laurent series for f which converges for $1<|z|<2$.

(c) $2<|z|<\infty$



Since $|z|>2$, we have

$$\left|\frac{2}{z}\right|<1 \qquad \text{— (i)}$$

Moreover $|z|>2 \Rightarrow |z|>1$

$\therefore$ we have $\left|\frac{1}{z}\right|<1$ (ii)

Keeping (i) and (ii) in mind, we can write f as

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{z\left(1-2/z\right)} \qquad \text{— (D)}$$

~~Note~~ Now,

$$\frac{1}{1-\frac{1}{z}} = 1+\frac{1}{z}+\frac{1}{z^2}+\cdots \qquad \text{when } \left|\frac{1}{z}\right|<1$$

i.e. $|z|>1$

and

$$\frac{1}{1-2/z} = 1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots \qquad \text{when } \left|\frac{2}{z}\right|<1$$

i.e. $|z|>2$

Using above in (D), we get

$$f(z) = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) - \frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\frac{2^3}{z^3}+\cdots\right)$$

when $2<|z|<\infty$

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

when $2<|z|<\infty$

Replacing 'n' by '$n-1$' in the last expression, we get

$$f(z) = \sum_{n=1} \frac{1-2^{n-1}}{z^n} \qquad (2 < |z| < \infty).$$

Q2
Page 205.

Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right]$$

$$(0 < |z+1| < \infty).$$

Soln.

Since e^z is an entire function

$\therefore$ The Taylor series for e^z about $z=-1$ is given by

$$e^z = \sum_{n=0}^{\infty} a_n (z-(-1))^n, \quad \text{where } a_n = \left.\frac{d^n e^z}{dz^n}\right|_{\text{at } z=-1}$$

$$= \left. e^z \right|_{\text{at } z=-1}$$

$$= e^{-1}. \quad \forall n.$$

$$\therefore e^z = \sum_{n=0}^{\infty} e^{-1} (z+1)^n$$

$$= e^{-1} + e^{-1}(z+1) + e^{-1}\frac{(z+1)^2}{2!} + \cdots$$

$|z+1| < \infty.$

Now, since $f(z) = \dfrac{e^z}{(z+1)^2}$ is analytic everywhere except at $z=-1$

$\therefore$ Laurent Series expression for $f(z)$ in the domain $0 < |z+1| < \infty$ is given by

$$f(z) = \frac{e^z}{(z+1)^2} = \frac{1}{e(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!}$$

$$= \frac{1}{e(z+1)^2}\left[1 + \frac{(z+1)}{1!} + \frac{(z+1)^2}{2!} + \cdots\right]$$

$$= \frac{1}{e}\left[\frac{1}{(z+1)^2} + \frac{1}{(z+1)} + \frac{1}{2!} + \frac{(z+1)}{3!} + \cdots\right]$$

$$= \frac{1}{e}\left[\frac{1}{(z+1)^2} + \frac{1}{(z+1)} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!}\right].$$

Q3. Find ~~the~~ a representation for the function

$$f(z) = \frac{1}{1+z}$$

in negative powers of z that is valid for $1 < |z| < \infty$

Hint: Write f as

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

Q4. Give two Laurent series expansions in powers of z for the function:

$$f(z) = \frac{1}{z^2(1-z)}$$

and specify the regions in which those expansions are valid.

Hint: ~~z=0, z=1 are sing~~ f is not analytic at $z=0, z=1$.

Region (1) $\quad 0 < |z| < 1$

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}\left(\sum_{n=0}^{\infty} z^n\right)$$

Region (2) $\quad |z| > 1$

$$f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2 \cdot z\left(1-\frac{1}{z}\right)} = \frac{1}{z^3}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

Q5. Represent the function $f(z) = \dfrac{z+1}{z-1}$

a) by its Maclaurin Series and state where the representation is valid

b) by its Laurent Series in the domain $1 < |z| < \infty$.

Hint: Write $f(z) = \frac{z+1}{z-1} = 1 + \frac{2}{z-1}$

a) $\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n \qquad (|z|<1)$

$\therefore f(z) = 1 - 2\sum_{n=0}^{\infty} z^n \qquad (|z|<1)$

b) The function f is analytic in the domain $1<|z|<\infty$

$$f(z) = 1 + \frac{2}{z-1} = 1 + \frac{2}{z\left(1-\frac{1}{z}\right)}$$

$$= 1 + 2\sum_{n=1}^{\infty} \frac{1}{z^n} \quad \text{when } |z|>1.$$

Q6. Show that when $0<|z-1|<2$

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

Hint: $f(z) = \frac{z}{(z-1)(z-3)}$ has two singularities at $z=1, z=3$

$\therefore$ It has Laurent series expansion in $0<|z-1|<2$.

Now, $f(z) = \frac{z}{(z-1)(z-3)} = \frac{(z-3)+3}{(z-1)(z-3)} = \frac{1}{(z-1)} + \frac{3}{(z-1)(z-3)}$ — (*)

Consider, $\frac{3}{(z-1)(z-2)} = \frac{-3}{2(z-1)\left[1-\frac{(z-1)}{2}\right]}$

$$= \frac{-3}{2(z-1)}\left[1-\frac{(z-1)}{2}\right]^{-1}$$

$$= \frac{-3}{2(z-1)}\left(\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n\right) \quad \left(\because 0<|z-1|<2 \quad \therefore \left|\frac{z-1}{2}\right|<1\right)$$

Use above in (*).

Q7 Write the two Laurent series in powers of z that represent the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

Hint $z=0, z=\pm i$ are singular points of f.

Note that $|\pm i| = 1$.

We can consider the following domains:

(a) $0<|z|<1$ (b) $1<|z|<\infty$.

a) $$f(z) = \frac{1}{z(1+z^2)}$$

~~Since~~ $0<|z|<1 \Rightarrow 0<|z^2|<1$

$$\therefore \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n (z^2)^n \quad \text{when } |z^2|<1$$

$$\Rightarrow \frac{1}{z(1+z^2)} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1} \quad \text{when } 0<|z^2|<1.$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty} (-1)^{n+1} z^{2n+1}$$

b) Since $1<|z|<\infty$

$1<|z^2|<\infty$

$$\Rightarrow \left|\frac{1}{z^2}\right|<1$$

Now, $$\frac{1}{1+z^2} = \frac{1}{z^2\left(1+\frac{1}{z^2}\right)}$$

$$\therefore f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z^3\left(1+\frac{1}{z^2}\right)}$$

$$\frac{1}{1+\frac{1}{z^2}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z^2}\right)^n \quad \text{when } \left|\frac{1}{z^2}\right|<1$$

i.e. when $|z|^2>1$

i.e. when $|z|>1$.

$$\therefore f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+3}} \quad \text{when } 1<|z|<\infty.$$