Chap-7

G·E-2

Orthogonality

Defn :- A set $S = \{v_1, v_2 \cdots v_k\}$ of k distinct vectors of $\mathbb{R}^n$ is said to be an orthogonal set of vectors if and only if every pair of distinct vectors in this set is orthogonal

i.e $v_i \cdot v_j = 0 \;\forall\; i \neq j$ {i.e dot product of any two vectors is zero}

Ex $S = \{(1,0,-1), (1,\sqrt{2},1), (1,-\sqrt{2},1)\}$

$(1,0,-1)\cdot(1,\sqrt{3},1) = 1+0-1 = 0$

$(1,0,-1)\cdot(1,-\sqrt{2},1) = 1+0-1 = 0$

$(1,\sqrt{2},1)\cdot(1,-\sqrt{2},1) = 1-2+1 = 0$

$\Rightarrow$ S is orthogonal set.

Defn :- Orthonormal Vector

A set $S = \{v_1, v_2 \cdots v_k\}$ of k distinct vectors of $\mathbb{R}^n$ is said to be an orthonormal set of vectors if and only if it is orthogonal set and all vectors are unit vectors

Theorem :- ~~orthogahality~~ orthogonality $\Rightarrow$ linear independence

Let $T = \{v_1, v_2 \cdots v_k\}$ be an orthogonal set of non-zero vectors in $\mathbb{R}^n$. Then T is linearly independent.

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Proof :- let $a_1 v_1 + a_2 v_2 + \cdots a_k v_k = 0$
for some $a_1, a_2 \cdots a_k \in \mathbb{R}$

Claim :- $a_1 = a_2 = \cdots = a_k = 0$

we know

$(a_1 v_1 + a_2 v_2 + \cdots a_k v_k) \cdot v_i = 0$ {any v_i} it is orthogonal

$\Rightarrow a_1 (v_1 \cdot v_i) + a_2 (v_2 \cdot v_i) + \cdots a_k (v_k \cdot v_i) = 0$

$\Rightarrow$ ~~$a_i (a_i$~~ $a_i (v_i \cdot v_i) = 0$ { $\because v_i \cdot v_i \neq 0$ as v_i are non-zero vector

$\Rightarrow a_i = 0$

Def^n :- Orthogonal and orthonormal Basis

A Basis B for a subspace W of $\mathbb{R}^n$ is said to be an orthogonal basis for W if and only if B is an orthogonal set.

If B is orthonormal Basis for W if and only if B is orthonormal set of

Corollary :- If B is an orthonormal set of n-non-zero vectors in $\mathbb{R}^n$, then B is an orthogonal Basis for $\mathbb{R}^n$. Similarly, if B is an orthonormal set of n vectors in $\mathbb{R}^n$ then B is an orthonormal Basis for $\mathbb{R}^n$.

Proof :- By above theorem
orthogonal set $\Rightarrow$ ~~ort~~ Linearly independent

$\Rightarrow$ B is Basis for $\mathbb{R}^n$

lly if B is orthonormal so it is orthogonal $\Rightarrow$ linearly independent

$\Rightarrow$ B is Basis for $\mathbb{R}^n$.

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Theorem :- Suppose that $B = \{v_1, v_2 \cdots v_n\}$ is a non-empty ordered orthogonal basis for $\mathbb{R}^n$ and v be any vector in $\mathbb{R}^n$. then

$$v = \underbrace{\frac{v \cdot v_1}{\|v_1\|^2}}_{a_1} v_1 + \underbrace{\frac{v \cdot v_2}{\|v_2\|^2}}_{a_2} v_2 + \cdots + \underbrace{\frac{v \cdot v_n}{\|v_n\|^2}}_{a_n} v_n$$

and the coordinatization of v with respect to B is

$$[v]_B = \cancel{\frac{v \cdot v_1}{\|v_1\|^2}} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \frac{v \cdot v_1}{\|v_1\|^2} \\ \frac{v \cdot v_2}{\|v_2\|^2} \\ \vdots \\ \frac{v \cdot v_n}{\|v_n\|^2} \end{bmatrix}$$

Proof :- $\because B = \{v_1, \cdots v_n\}$ is an ordered basis for $\mathbb{R}^n$ & $v \in \mathbb{R}^n \Rightarrow$ there exist $a_1, \cdots a_n$ scalars such that

$$v = a_1 v_1 + a_2 v_2 \cdots a_n v_n$$

{definition of basis and linear combination}

Claim :- $a_i = \dfrac{v \cdot v_i}{\|v_i\|^2}$

$$v \cdot v_i = (a_1 v_1 + a_2 v_2 \cdots a_i v_i + \cdots a_n v_n) \cdot v_i$$

$$= a_1(v_1 \cdot v_i) + a_2(v_2 \cdot v_i) + \cdots + a_i(v_i \cdot v_i) + \cdots + a_n(v_n \cdot v_i)$$

$\because B$ is orthogonal Basis

$\Rightarrow v_i \cdot v_j = 0$ for $i \neq j$

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$\Rightarrow v \cdot v_i = a_1(0) + a_2(0) + \cdots + a_i(\|v_i\|^2) + \cdots a_n(0)$

$\Rightarrow a_i = \dfrac{v \cdot v_i}{\|v_i\|^2}$ $\{ v_i \cdot v_i = \|v_i\|^2 \}$

Corollary :- suppose $B = \{v_1 \cdots v_n\}$ is an ordered orthonormal basis for $\mathbb{R}^n$, and v be any vector in $\mathbb{R}^n$. Then
$$v = (v \cdot v_1) v_1 + \cdots (v \cdot v_n) v_n$$

Proof :- by above theorem
$$v = \frac{v \cdot v_1}{\|v_1\|^2} v_1 + \cdots \frac{v \cdot v_n}{\|v_n\|^2} v_n$$

$\because$ B is orthonormal basis $\Rightarrow \|v_i\|^2 = 1$

$\Rightarrow v = (v \cdot v_1) v_1 + \cdots (v \cdot v_n) v_n$

hence $[v]_B = \begin{bmatrix} v \cdot v_1 \\ \vdots \\ v \cdot v_n \end{bmatrix}$

Ques :- let $B = \left\{ v_1 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right), v_2 = \left(\frac{-1}{3\sqrt{3}}, \frac{4}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}}\right), v_3 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \right\}$

is an orthonormal basis for $\mathbb{R}^3$ & find $[v]_B$ for $v = \{-1, 5, 3\}$

Sol? verify ~~find~~ $\|v_1\|^2 = \|v_2\|^2 = \|v_3\|^2 = 1$

& $v_1 \cdot v_2 = v_2 \cdot v_3 = v_1 v_3 = 0$

$\Rightarrow$ orthonormal

then use ~~for~~ above corollary

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gmp

Gram-schmidt Process :- Let $\{w_1, w_2, \ldots w_k\}$ be linearly independent set of vectors in $\mathbb{R}^n$. The Gram-schmidt Process contructs a new set $\{v_1, v_2 \ldots v_k\}$ of vectors such that $\{v_1, v_2 \ldots v_k\}$ become an orthogonal basis for span $\{w_1, w_2 \ldots w_k\}$.

* Following are the steps in Gram-~~sm~~ schmidt Process

Step ① let $v_1 = w_1$

Step ② let $v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{\|v_1\|^2}\right) v_1$ $\quad v_2 = w_2 - \text{proj}_{v_1} w_2$

Step ③ let $v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{w_3 - v_2}{\|v_2\|^2}\right) v_2$

Continue this process upto v_k. We will get the ~~set~~ set $\{v_1, v_2 \ldots v_k\}$.

Theorem :- Let $B = \{w_1, w_2 \ldots w_k\}$ be a basis for a subspace W of $\mathbb{R}^n$. Then the set $T = \{v_1, v_2 \ldots v_k\}$ obtained by applying the Gram-schmidt process to B is an orthogonal basis for W. {by definition of Gram-schmidt Process}

Ex :- let $B = \{(2,1,0,-1), (1,0,2,-1), (0,-2,1,0)\}$ ~~of~~ subset of $\mathbb{R}^4$. verify B is a linearly independent set. find the orthogonal basis for ~~to~~ $\mathbb{R}^4$.

B is not orthogonal basis as

$(2,1,0,-1) \cdot (1,0,2,-1) \neq 0$

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Prove B is linearly independent (verify) 5

Now

Using Gram-Schmidt process

let $v_1 = w_1 = (2,1,0,-1)$

$$v_2 = w_2 - \left(\frac{w_2 \cdot v_1}{\|v_1\|^2}\right) v_1$$

$$= (1,0,2,-1) - \frac{(1,0,2,-1)\cdot(2,1,0,-1)}{\|(2,1,0,-1)\|^2}(2,1,0,-1)$$

$$= (1,0,2,-1) - \frac{3}{6}(2,1,0,-1)$$

$$= (1,0,2,-1) - (1, 1/2, 0, -1/2)$$

$$= (0, -1/2, 2, -1/2)$$

Ily

$$v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{\|v_1\|^2}\right)v_1 - \left(\frac{w_3 \cdot v_2}{\|v_2\|}\right)v_2$$

Ily we can find $v_4 = w_4 - \left(\frac{w_4 \cdot v_1}{\|v_1\|^2}\right)v_1 - \left(\frac{w_4 \cdot v_2}{\|v_2\|^2}\right)v_2 - \left(\frac{w_4 \cdot v_3}{\|v_3\|^2}\right)v_3$

theorem :- let W be a subspace of $\mathbb{R}^n$. Then any orthogonal set of non-zero vectors in W can be enlarged to an orthogonal basis for W.

Ily for orthonormal

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Def^n (orthogonal complement)

Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W, denoted by $W^\perp$, is the set of all vectors of $\mathbb{R}^n$ that are orthogonal to every vector in W.

$$W^\perp = \{x \in \mathbb{R}^n : x \cdot w = 0 \text{ for all } w \in W\}$$

Theorem :- If W is a subspace of $\mathbb{R}^n$, then ~~$v \in W^\perp$~~ $v \in W^\perp$ if and only if v is orthogonal to every vector in a spanning set for W.

Proof :- let $S = \{w_1, w_2, \ldots, w_k\}$ be spanning set for W i.e. $W = \text{span}(S)$

let $v \in W^\perp$

Claim :- $v \cdot w_i = 0$ for all $w_i \in \text{span}(S) = W$

by definition, $v \cdot w = 0$ for all $w \in W$

$\Rightarrow$ in particular
$$v \cdot w_i = 0$$

Conversely

let ~~$v \in$~~ $v \cdot w_i = 0$ for all $w_i \in \text{span}(S) = W$

$\Rightarrow$ for any $w \in W$ there exist $a_1, a_2, \ldots, a_k$ such that

$$w = a_1 w_1 + a_2 w_2 + \ldots + a_k w_k.$$

Now $v \cdot w = a_1\, v \cdot w_1 + a_2\, v \cdot w_2 + \ldots + a_k\, v \cdot w_k$
$= 0$

$\Rightarrow v \in W^\perp$

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Ex let $W = \{(a, 0, c) ; a, c \in \mathbb{R}\}$ subspace of $\mathbb{R}^3$ and let W is spanned by the set $\{(1,0,0), (0,0,1)\}$ find $W^\perp$

let $(x, y, z) \in W^\perp$

$\Rightarrow (x, y, z)\cdot(1, 0, 0) = 0$

& $(x, y, z)\cdot(0, 0, 1) = 0$

$\Rightarrow x = 0, z = 0$

$\Rightarrow W^\perp = \{(0, y, 0) ; y \in \mathbb{R}\} = \text{span}\{(0,1,0)\}$

Theorem :- Let W be a subspace of $\mathbb{R}^n$. Then $W^\perp$ is a subspace of $\mathbb{R}^n$ and $W \cap W^\perp = \{0\}$.

Proof :- clearly $0 \in W^\perp$ because 0 is orthogonal to each vector

let $x_1, x_2 \in W^\perp$, and $w \in W$

$(x_1 + x_2)\cdot w = x_1 \cdot w + x_2 \cdot w = 0$

$\Rightarrow x_1 + x_2 \in W^\perp$

Now $(cx)\cdot w = c(x \cdot w) = 0$

$\Rightarrow cx \in W^\perp$

$\Rightarrow W^\perp$ is subspace of $\mathbb{R}^n$

Now let $w \in W \cap W^\perp$

$\Rightarrow w \in W$ & $w \in W^\perp$

$\Rightarrow w \cdot w = 0$

$\Rightarrow w = 0$

$\Rightarrow$ $W \cap W^\perp = \{0\}$

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*

Theorem:- Let W be a subspace of $\mathbb{R}^n$. Let $\{v_1, v_2 \dots v_k\}$ be an orthogonal basis for W contained in an orthogonal base $\{v_1, v_2 \dots v_k, v_{k+1} \dots v_n\}$ for $\mathbb{R}^n$. Then $\{v_{k+1}, v_{k+2} \dots v_n\}$ is orthogonal basis for $W^\perp$

~~Theorem~~ :-
Corollary:- Let W be a subspace of $\mathbb{R}^n$. Then
$\dim(W) + \dim(W^\perp) = n = \dim(\mathbb{R}^n)$

Proof:- {by the theorem that let $B = \{w_1, w_2 \dots w_k\}$ be a basis for subspace W of $\mathbb{R}^n$. the the set $T = \{v_1, v_2 \dots v_k\}$ obtained by applying the gram-~~sm~~ schmidt process to B is an orthogonal basis.}

$\Rightarrow$ W has orthogonal basis
let $\{v_1, v_2 \dots v_k\}$

Claim:- $\{v_1, v_2 \dots v_k\}$ ~~of~~ is orthogonal set of non-zero vectors in $\mathbb{R}^n$.

Now by theorem 7.6 we can extend this set to an orthogonal basis $\{v_1, v_2 \dots v_k, v_{k+1} \dots v_n\}$ for $\mathbb{R}^n$

using above theorem (*)
$\{v_{k+1} \dots v_n\}$ is orthogonal basis for $W^\perp$

$\Rightarrow \dim(W^\perp) = n - k$
$\Rightarrow \dim(W) + \dim W^\perp = k + n - k = n = \dim(\mathbb{R}^n)$

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Ex consider a subspace $W = span(\{(1,2,3)\})$ (10)

of $\mathbb{R}^3$, find $W^{\perp}$.

Sol^n $\because$ $W = span(\{(1,2,3)\})$

$\Rightarrow dim(W) = 1$

Now let $(x,y,z) \in W^{\perp}$

$\Rightarrow (x,y,z).(1,2,3) = 0$

$\Rightarrow W^{\perp} = \{(x,y,z) \in \mathbb{R}^3 ; x+2y+3z=0\}$

Corollary :- let W be an subspace of $\mathbb{R}^n$

Then $(W^{\perp})^{\perp} = W$.

Proof :- we will show $W \subseteq (W^{\perp})^{\perp}$ and $(W^{\perp})^{\perp} \subseteq W$

clearly $W \subseteq (W^{\perp})^{\perp}$

Now claim $W = (W^{\perp})^{\perp}$

i.e we will show $dim(W) = dim(W^{\perp})^{\perp}$

by previous theorem i.e $dim(W) + dim\, W^{\perp} = n$

$\Rightarrow$ lly $\cancel{dim\,(W^{\perp})} + dim\,(W^{\perp})^{\perp} = \cancel{n} = dim(W) + \cancel{dim(W^{\perp})}$

$\Rightarrow dim\,(W^{\perp})^{\perp} = dim\, W$

$\Rightarrow W = (W^{\perp})^{\perp}$

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Ex For the subspace $W = \{(x, y, z) \in \mathbb{R}^3 ; 3x - y + 4z = 0\}$ (11)
of $\mathbb{R}^3$, Find the orthogonal complement $W^\perp$ and verify that $\dim(W) + \dim(W^\perp) = \dim(\mathbb{R}^3)$

Sol^n:-

$$W = \{(x, y, z) \in \mathbb{R}^3 ; 3x - y + 4z = 0\}$$

$$= (x, y, z) \cdot (3, -1, 4) = 0$$

$\Rightarrow (x, y, z)$ orthogonal to $(3, -1, 4)$

$$\Rightarrow W^\perp = \{(3, -1, 4)\}$$

also $W = \text{span}\{(1, 3, 0), (-4, 0, 3)\}$

{ Hint:- take $x = 1, z = 0$ you get $(1, 3, 0)$
take $y = 0, z = 3$ you get $(1, 3, 0)$

$\Rightarrow \dim(W) = 2$
and $\dim(W^\perp) = 1$
$\Rightarrow \dim(W) + \dim(W^\perp) = 3$

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