**Ex. 15.** *Find the numerical solution of*

$\frac{dy}{dx} = x + y$, *from* $x = 0$ *to* $0{\cdot}2$ *by Euler's method and Modified Euler's method, with the initial condition* $x_0 = 0, y_0 = 1$.

**Solution : (*i*) By Euler's Method**

$y_{n+1} = y_n + hf(x_n, y_n), n = 0, 1, 2, ....$

Taking $h = 0{\cdot}05$, the approximate value of $y_1$ is

$$y_1 = y_0 + hf(x_0, y_0) = y_0 + h(x_0 + y_0)$$
$$= 1 + 0.05(0 + 1)$$
$$= 1{\cdot}0500$$

The approximate value of $y_2$ is $y_2 = y_1 + hf(x_1, y_1) = y_1 + hf(x_1 + y_1)$.

where $x_1 = x_0 + h = 0 + 0{\cdot}05 = 0{\cdot}05$.

∴ Substituting values

$$y_2 = 1{\cdot}05 + 0{\cdot}05(0{\cdot}05 + 1{\cdot}05)$$
$$= 1{\cdot}1050$$

Similarly,

$$y_3 = y_2 + hf(x_2, y_2) \text{ with } x_2 = x_0 + 2h$$
$$= 1{\cdot}105 + 0{\cdot}05(0{\cdot}10 + 1{\cdot}105)$$
$$= 1{\cdot}16525$$

and $$y_4 = y_3 + hf(x_3, y_3) \text{ with } x_3 = x_0 + 3h$$
$$= 1{\cdot}16525 + 0{\cdot}05(0{\cdot}15 + 1{\cdot}16525)$$
$$= 1{\cdot}2310$$

These results are shown in the following table :

| $n$ | $x_n$ | $y_n$ | $\frac{dy}{dx} = f(x, y) = x + y$ |
|---|---|---|---|
| 0 | 0·00 | $y_0 = 1{\cdot}0000$ | 1·0000 |
| 1 | 0·05 | $y_1 = 1{\cdot}0500$ | 1·1000 |
| 2 | 0·10 | $y_2 = 1{\cdot}1050$ | 1·2050 |
| 3 | 0·15 | $y_3 = 1{\cdot}1652$ | 1·3153 |
| 4 | 2·0 | $y_4 = 1{\cdot}2310$ | 1·4310 |

(*ii*) **Modified Euler's method : According to this approximation**

$$y_{n+1}^{(r+1)} = y_n + \frac{1}{2}h\,[f(x_0, y_0) + f(x_n, y_n^{(r)})] \qquad ...(1)$$

The value of $r$ for a given $n$ is repeated until no significant change occurs in $y_{n+1}^{(r+1)}$

**For First Interval**

Taking $h = 0{\cdot}05$, we have

$$y_1^{(1)} = y_0 + hf(x_0, y_0) = 1 + 0{\cdot}05(0 + 1) = 1{\cdot}05$$

The second approximation to $y_1$ is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})] = 1 + \frac{0\cdot05}{2}[1 + 0\cdot05 + 1\cdot05] = 1\cdot0525$$

The third approximation to $y_1$ is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})] = 1 + \frac{0\cdot05}{2}[1 + 0\cdot05 + 1\cdot0525] = 1\cdot05256$$

The fourth approximation to $y_1$ is

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})] = 1 + \frac{0\cdot05}{2}[1 + 0\cdot05 + 1\cdot05256] = 1\cdot05256$$

Clearly $y_1^{(4)} = y_1^{(3)}$, therefore $y_1 = \mathbf{1\cdot05256}$ ...(2)

**For Second Interval** : $f(x_1, y_1) = x_1 + y_1 = 0\cdot05 + 1\cdot0526 = 1\cdot1026$

We have $y_2^{(1)} = y_1 + hf(x_1, y_1) = 1\cdot0526 + 0\cdot05 \times 1\cdot1026 = 1\cdot1077$

The second approximate value of $y_2$ is

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})] = 1\cdot0526 + \frac{0\cdot05}{2}[1\cdot1026 + 0\cdot1 + 1\cdot1077]$$

$$= 1\cdot1104$$

The third approximate value of $y_2$ is

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 1\cdot0526 + \frac{0\cdot05}{2}[1\cdot1026 + 0\cdot1 + 1\cdot1104] = 1\cdot1104$$

Clearly $y_2^{(3)} = y_2^{(2)}$. Hence we take

$$y_2 = 1\cdot1104$$

**For third Interval** : $f(x_2, y_2) = x_2 + y_2 = 2 \times 0\cdot05 + 1\cdot1104 = 1\cdot2104$

The first approximation to $y_3$ is

$$y_3^{(1)} = y_2 + hf(x_2, y_2) = 1\cdot1104 + 0\cdot05\,[1\cdot2104] = 1\cdot1709$$

The second approximation to $y_3$ is

$$y_3^{(2)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(1)})]$$

$$\simeq 1\cdot1104 + \frac{0\cdot05}{2}[1\cdot2104 + 3 \times 0\cdot05 + 1\cdot1709]$$

$$\simeq 1\cdot1737$$

The third approximation to $y_3$ is

$$y_3^{(3)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(2)})]$$

$$= 1\cdot1104 + \frac{0\cdot05}{2}[1\cdot2104 + (3 \times 0\cdot05 + 1\cdot1737)]$$

$$\simeq 1\cdot1737$$

As $y_3^{(3)} = y_2^{(2)} \Rightarrow y_3 = 1\cdot1737$ ...(4)

**For fourth Interval :** $[f(x_3, y_3) = 3 \times 0{\cdot}05 + 1{\cdot}1737 = 1{\cdot}3237]$

The first approximation to $y_4$ is

$$y_4^{(1)} = y_3 + hf(x_3, y_3) = 1{\cdot}1737 + 0{\cdot}05 \times 1{\cdot}3237 = 1{\cdot}2399$$

The second approximation to $y_4$ is

$$y_4^{(2)} = y_3 + \frac{h}{2}[f(x_3, y_3) + f(x_4, y_4^{(1)})]$$

$$= 1{\cdot}1737 + \frac{0{\cdot}05}{2}[1{\cdot}3237 + 4 \times 0{\cdot}05 + 1{\cdot}2399]$$

$$= 1{\cdot}2428$$

The third approximation to $y_4$ is

$$y_4^{(3)} = y_3 + \frac{h}{2}[f(x_3, y_3) + f(x_4, y_4^{(2)})]$$

$$= 1{\cdot}1737 + \frac{0{\cdot}05}{2}[1{\cdot}3237 + 4 \times 0{\cdot}05 + 1{\cdot}2428]$$

$$= 1{\cdot}2428$$

Clearly $y_4^{(3)} = y_4^{(2)} \Rightarrow y_4 = 1{\cdot}2428$

The results deduced are tabulated as

| $n$ | $x_n$ | $y_n$ | $dy/dx = (x + y)$ |
|---|---|---|---|
| $n = 0$ | 0·00 | 1·0000 | 1·0000 |
| 1 | 0·05 | 1·0526 | 1·1026 |
| 2 | 0·10 | 1·1104 | 1·2104 |
| 3 | 0·15 | 1·1737 | 1·3237 |
| 4 | 0·20 | 1·2428 | 1·4428 |

(c) **Taylor-Series Method** : Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \text{ with } y = y_0,\ x = x_0 \qquad \ldots(1)$$

This may be expressed as

$$dy = f(x,y)\, dx$$

Integrating, $y = \int f(x, y)\, dx = F(x)$ ...(2)

Expanding $F(x)$ in the neighbourhood of $x_0$ by Taylor's expansion, we get

$$y = F(x) = f[x_0 + (x - x_0)]$$

$$= F(x_0) + (x - x_0) F'(x_0) + \frac{(x - x_0)^2}{2!} F''(x_0) + \cdots \qquad \ldots(3)$$

$$= y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots$$

$$= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots \qquad \ldots(4)$$

where $x = x_0 + h$

Equation (4) is convergent in $x$ for $x_0 \le x \le x_n$ where $x_n = x_0 + nh$, hence $y_0''$, $y_0'''$ etc. may be computed from (1) as

**Ex. 18.** *Use Runge-Kutta method to solve the equation*

$$\frac{dy}{dx} = x + y \text{ with initial conditions } x_0 = 0,\ y_0 = 1$$

*from* $x = 0$ *to* $x = 0{\cdot}4$ *with interval* $h = 0{\cdot}1$.

**Solution :** Given equation is $\frac{dy}{dx} = x + y$ with $x_0 = 0$, $y_0 = 1$

*i.e.* $y'(x) = f(x)$ where $f(x) = x + y$ ...(1)

We have for the first interval ($x_0 = 0$)

$$A_0 = hf(x_0, y_0) = 0{\cdot}1\ (x_0 + y_0) = 0{\cdot}1\ (0 + 1) = 0{\cdot}1$$

$$B_0 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{A_0}{2}\right) = 0{\cdot}1\left[0 + 0{\cdot}05 + 1 + \frac{0{\cdot}1}{2}\right] = 0{\cdot}11$$

$$C_0 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{B_0}{2}\right) = 0{\cdot}1\left[0 + 0{\cdot}05 + 1 + \frac{0{\cdot}11}{2}\right] = 0{\cdot}1105$$

$$D_0 = hf(x_0 + h, y_0 + C_0) = 0{\cdot}1\ [0 + 0{\cdot}1 + 1 + 0{\cdot}1105] = 0{\cdot}12105$$

$$\therefore\quad y_1 = y_0 + \frac{1}{6}[A_0 + 2B_0 + 2C_0 + D_0]$$

$$= 1 + \frac{1}{6}[0{\cdot}1 + 2 \times 0{\cdot}11 + 2 \times 0{\cdot}1105 + 0{\cdot}12105] = 1{\cdot}11034$$

For *the second interval* ($x_1 = x_0 + h = 0 + 0{\cdot}1 = 0{\cdot}1$)

$$A_1 = hf(x_1, y_1) = 0{\cdot}1\ (x_1 + y_1) = 0{\cdot}1\ (0{\cdot}1 + 1{\cdot}11034) = 0{\cdot}121034$$

$$B_1 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{A_1}{2}\right) = 0{\cdot}1\left[0{\cdot}1 + \frac{0{\cdot}1}{2} + 1{\cdot}11034 + \frac{0{\cdot}120134}{2}\right] = 0{\cdot}13208$$

$$C_1 = hf\left[x_1 + \frac{h}{2}, y_1 + \frac{B_1}{2}\right] = 0{\cdot}1\left[0{\cdot}1 + \frac{0{\cdot}1}{2} + 1{\cdot}11034 + \frac{0{\cdot}13208}{2}\right] = 0{\cdot}13263$$

$$D_1 = hf(x_1 + h, y_1 + C_1) = 0{\cdot}1\ [0{\cdot}1 + 0{\cdot}1 + 1{\cdot}11034 + 0{\cdot}13263] = 0{\cdot}14429$$

$$\therefore\quad y_2 = y_1 + \frac{1}{6}[A_1 + 2B_1 + 2C_1 + D_1]$$

$$= 1{\cdot}11034 + \frac{1}{6}[0{\cdot}121034 + 2 \times 0{\cdot}13208 + 2 \times 0{\cdot}13263 + 0{\cdot}14429] = 1{\cdot}2428$$

For the *third interval* ($x_3 = x_0 + 2h = 0 + 2 \times 0{\cdot}1 = 0{\cdot}2$)

$$A_2 = hf(x_2, y_2) = 0{\cdot}1\ [0{\cdot}2 + 1{\cdot}2428] = 0{\cdot}14428$$

$$B_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{A_2}{2}\right) = 0{\cdot}1\left[0{\cdot}2 + \frac{0{\cdot}1}{2} + 1{\cdot}2428 + \frac{0{\cdot}14428}{2}\right] = 0{\cdot}15649$$

$$C_2 = hf\left[x_2 + \frac{h}{2}, y_2 + \frac{B_2}{2}\right] = 0{\cdot}1\left[0{\cdot}2 + \frac{0{\cdot}1}{2} + 1{\cdot}2428 + \frac{0{\cdot}15649}{2}\right] = 0{\cdot}15710$$

$$D_2 = hf(x_2 + h, y_2 + C_2) = 0{\cdot}1\ [0{\cdot}2 + 0{\cdot}1 + 1{\cdot}2428 + 0{\cdot}15710] = 0{\cdot}16999$$

$$\therefore\quad y_3 = y_2 + \frac{1}{6}[A_2 + 2B_2 + 2C_2 + D_2]$$

$$= 1{\cdot}2428 + \frac{1}{6}[0{\cdot}14428 + 2 \times 0{\cdot}15649 + 2 \times 0{\cdot}15710 + 0{\cdot}16999] = 1{\cdot}3997$$

For the *Fourth interval* $x_3 = x_0 + 3h = 0 + 3 \times 0{\cdot}1 = 0.3)$

$$A_3 = hf(x_3, y_3) = 0{\cdot}1\ (0{\cdot}3 + 1{\cdot}3997) = 0{\cdot}16997)$$

$$B_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{A_3}{2}\right) = 0{\cdot}1\left(0{\cdot}3 + \frac{0{\cdot}1}{2} + 1{\cdot}3997 + \frac{0{\cdot}16997}{2}\right) = 0{\cdot}18347$$

$$C_3 = hf\left(x_3 + \frac{h}{2}, y_3 + \frac{B_3}{2}\right) = 0{\cdot}1\left[0{\cdot}3 + \frac{0{\cdot}1}{2} + 1{\cdot}3997 + \frac{0{\cdot}18347}{2}\right] = 0{\cdot}18414$$

$$D_3 = hf(x_3 + h, y_3 + C_3) = 0{\cdot}1\ [0{\cdot}3 + 0{\cdot}1 + 1{\cdot}3997 + 0{\cdot}18414] = 0{\cdot}19838$$

$$\therefore \quad y_4 = y_3 + \frac{1}{6}(A_3 + 2B_3 + 2C_3 + D_3)$$

$$= 1{\cdot}3997 + \frac{1}{6}[0{\cdot}16997 + 2 \times 0{\cdot}18347 + 2 \times 0{\cdot}18414 + 0{\cdot}19838] = 1{\cdot}5836$$

**Ex. 19.** *Use Runge-Kutta method to find by(0.2) for the equation*

$$\frac{dy}{dx} = \frac{y - x}{y + x}, \quad y(0) = 1, \text{ take } h = 0{\cdot}2.$$

**Solution.** Given equation is

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

Here $x_0 = 0, y_0 = 1, h = 0{\cdot}2$

$$f(x, y) = \frac{y - x}{y + x}$$

$$f(x_0, y_0) = \frac{y_0 - x_0}{y_0 + x_0} = \frac{1 - 0}{1 + 0} = 1$$

Hence $A_0 = hf(x_0, y_0) = 0{\cdot}2 \times 1 = 0{\cdot}2$

$$B_0 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{A_0}{2}\right) = 0{\cdot}2\, f\left(0 + \frac{0{\cdot}2}{2},\ 1 + \frac{0{\cdot}2}{2}\right)$$

$$= 0{\cdot}2\, f(0{\cdot}1, 1{\cdot}1) = 0{\cdot}2\left[\frac{1{\cdot}1 - 0{\cdot}1}{1{\cdot}1 + 0{\cdot}1}\right]$$

$$= 0{\cdot}2 \times \left(\frac{1{\cdot}0}{1{\cdot}2}\right) = 0{\cdot}1667$$

$$C_0 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{B_0}{2}\right) = 0{\cdot}2f\left(0 + \frac{0{\cdot}2}{2}, 1 + \frac{0{\cdot}1667}{2}\right)$$

$$= 0{\cdot}2f(0{\cdot}1, 1{\cdot}0834)$$

$$= 0{\cdot}2\left[\frac{1{\cdot}0834 - 0{\cdot}1}{1{\cdot}0834 + 0{\cdot}1}\right] = (0{\cdot}2 \times \left(\frac{0{\cdot}9834}{1{\cdot}1834}\right) = 0{\cdot}1662$$

$$D_0 = hf(x_0 + h, y_0 + C_0)$$

$$= 0{\cdot}2f(0 + 0{\cdot}2, 1 + 0{\cdot}1662)$$

$$= 0{\cdot}2f(0{\cdot}2, 1{\cdot}1662) = 0{\cdot}2 \times \left[\frac{1{\cdot}1662 - 0{\cdot}2}{1{\cdot}1662 + 0{\cdot}2}\right]$$

$$= 0{\cdot}2 \times \left(\frac{0{\cdot}9662}{1{\cdot}3662}\right) = 1{\cdot}4144$$

$$\therefore \quad \Delta y = \frac{1}{6}(A_0 + 2B_0 + 2C_0 + D_0)$$

$$= \frac{1}{6}[0 \cdot 2 + 2 \times (0 \cdot 1667) + 2 \times (0 \cdot 1662) + 1 \cdot 4144]$$

$$= \frac{1}{6}[0 \cdot 2 + 0 \cdot 3334 + 0 \cdot 3324 + 1 \cdot 4144]$$

$$= \frac{1}{6}(2 \cdot 2802) = 0 \cdot 3800$$

$\therefore$ Value of $y(0 \cdot 2) = y + \Delta y = 1 + 0 \cdot 3800 =$ **1·3800**

## 14·7. Approximate Solution of Algebraic and Transcendental Equations

(*a*) **Newton-Raphson Method** : This provides the method of finding the roots of an equation.

Let given equation be $f(x) = 0$ ...(1)

Let its approximate solution be $x_0$ and exact solution $(x_0 + h)$ where $h$ is very small quantity. Then we have $f(x_0 + h) = 0$. ...(2)

Expanding this by Taylor's series about $x_0$, we find

$$f(x_0 + h) = f(x_0) + \frac{h}{1!}f'(x_0) + \frac{h^2}{2!}f''(x_0) \;.... + = 0 \qquad ...(3)$$

As $h$ is very small quantity therefore the terms containing $h^2$ and higher powers of $h$ may be neglected. Then (3) becomes

$$f(x_0 + h) = f(x_0) + hf'(x_0) = 0$$

This gives $h = -\dfrac{f(x_0)}{f'(x_0)}$, provided $x_0$ exists. Denoting $h$ by $h_1$, we have

$$h_1 = -\frac{f(x_0)}{f'(x_0)} \qquad ...(4)$$

This equation gives a value of $h$, which when added to $x_0$ would give better approximation to the root. Let this root be $x_1$. Then

$$x_1 = x_0 + h_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad ...(5)$$

Now using $x_1$ in place of $x_0$ and $h_2$ the new value of $h$, (in analogy with (3))

$$h_2 = -\frac{f(x_1)}{f'(x_1)} \qquad ...(6)$$

we get the new value of root $x_2$ (say) to the second approximation, so that

$$x_2 = x_1 + h_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad ...(7)$$

Continuing the process, we get successive values of $h$ as

$$h_3 = -\frac{f(x_2)}{f'(x_2)},\; h_4 = -\frac{f(x_3)}{f'(x_3)},......,\; h_n = \frac{f(x_{n-1})}{f'(x_{n-1})}$$

and the higher order approximated roots being

$$x_3 = x_2 + h_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$