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Note: ① For any abelian group G

$$aH = Ha \quad \forall a \in G, H \leq G$$

② For any non abelian group G

aH may or may not be the same as Ha.

Properties of Cosets:

Let H be a subgroup of G and let a and b belongs to G. Then

① $a \in aH$ and $b \in bH$

② $aH = H$ iff $a \in H$

③ $(ab)H = a(bH)$ and $H(ab) = (Ha)b$

④ $aH = bH$ iff $a \in bH$

⑤ $aH = bH$ or $aH \cap bH = \phi$

⑥ $aH = bH$ iff $a^{-1}b \in H$

7 $|aH| = |bH|$

8 $aH = Ha$ iff $H = aHa^{-1}$

9 aH is a subgroup of G iff $a \in H$.

Proof:

1 $a \in aH$ and $b \in bH$

$aH = \{ah \mid h \in H\}$

$\because H$ is the subgroup of $G \Rightarrow e \in H$

$\therefore ae \in aH \Rightarrow a \in aH$

Similarly $be \in bH \Rightarrow b \in bH$.

2 $aH = H$ iff $a \in H$

First, let $aH = H$

$\because a = ae$ and $e \in H$

$\Rightarrow a \in aH$

$\Rightarrow a \in H$ $\quad (\because aH = H)$

Conversely ($\Leftarrow$) let $a \in H$

TS $aH = H$. (TS $aH \subseteq H$ & $H \subseteq aH$)

$\because aH = \{ah \mid h \in H\} \subseteq H$ [$\because ah \in H$ $\forall a \in H, h \in H$ (closure property)]

$\Rightarrow aH \subseteq H$ —— ①

let $h \in H$

Now, $h = eh = (aa^{-1})h$

$= a(a^{-1}h)$ (Associativity)

$\in aH$. [$\because H \leq G$, $a \in H, h \in H$ $\Rightarrow a^{-1} \in H, h \in H$ $\Rightarrow a^{-1}h \in H$.]

$\therefore H \subseteq aH$ —— ②

from eqn ① & ②,

$aH = H$.

③ $(ab)H = a(bH)$ and $H(ab) = (Ha)b$

$\because (ab)h = a(bh)$ $\forall h \in H$ (Associativity)

$\Rightarrow (ab)H = a(bH)$

Similarily, $h(ab) = (ha)b \ \forall h \in H$ (Associativity)

$\Rightarrow H(ab) = (Ha)b$

④ $aH = bH$ iff $a \in bH$

First, let $aH = bH$

Now, $a = ae \in aH = bH \Rightarrow a \in bH$.

Conversely ($\Leftarrow$) let $a \in bH$ $\quad [bH = \{bh \mid h \in H\}]$

$\Rightarrow \exists h_1 \in H$ such that $a = bh_1$

Now, $aH = (bh_1)H = b(h_1 H)$ $\quad [h_1 H = H$ Property ②$]$

$= bH$

$\Rightarrow aH = bH$.

⑤ $aH = bH$ or $aH \cap bH = \phi$

let $c \in aH \cap bH$

TS $aH = bH$.

Property ④ $[aH = bH \Leftrightarrow a \in bH]$

$\because c \in aH \Rightarrow cH = aH$ — ③

$\because c \in bH$.

$\cdots \text{—— } (3)$

$\because c \in bH \Rightarrow cH = bH$ —— (4)

From eqn (3) & (4), $aH = bH$.

(6) $aH = bH$ iff $a^{-1}b \in H$

First let $aH = bH \Rightarrow a^{-1}aH = a^{-1}bH$

$\Rightarrow H = a^{-1}bH$

$\Rightarrow a^{-1}bH = H$

$\Rightarrow a^{-1}b \in H$ [Property (2)]

Conversely ($\Leftarrow$) let $a^{-1}b \in H$

$\Rightarrow a^{-1}bH = H$ [Property (2)]

$\Rightarrow a(a^{-1}bH) = aH$

$\Rightarrow aa^{-1}(bH) = aH$

$\Rightarrow bH = aH$

$\Rightarrow aH = bH$.

(7) $|aH| = |bH|$.

Define a function $f: aH \to bH$ such that

$$f(ah) = bh \quad \forall\, h \in H,$$
$$ah \in aH.$$

__One-one:__

Let $f(ah_1) = f(ah_2)$

$\Rightarrow bh_1 = bh_2$

$\Rightarrow b^{-1}bh_1 = b^{-1}bh_2$

$\Rightarrow h_1 = h_2$

$\Rightarrow ah_1 = ah_2$

$\therefore f$ is one-one.

__Onto:__

For every $bh \in bH$, there exist $ah \in aH$ such that $f(ah) = bh$.

$\therefore f$ is onto

Thus, f is one-one & onto