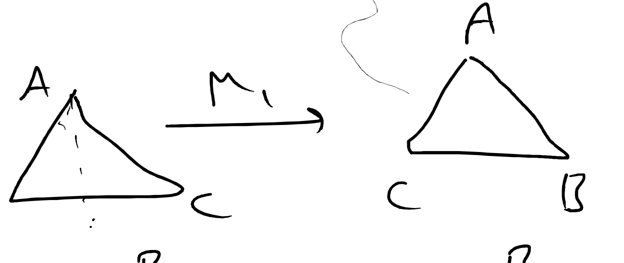
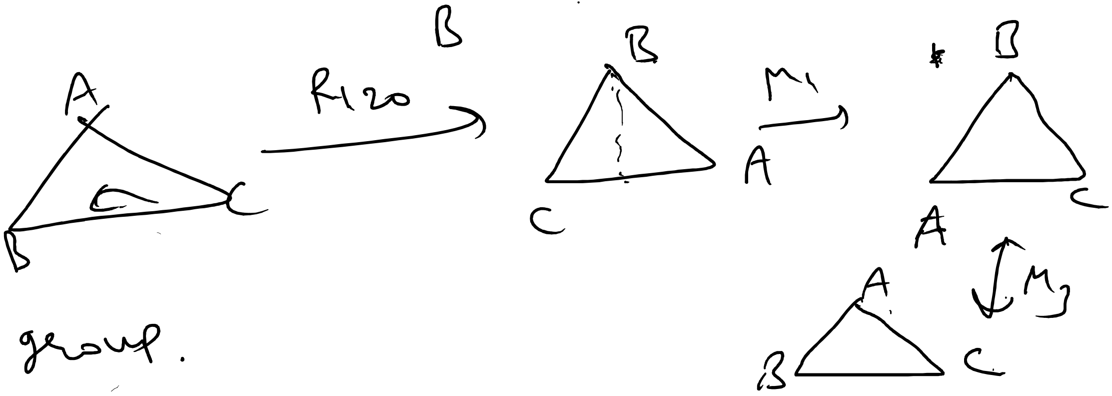
$R_0, R_{120}, R_{240}, M_A, M_B, M_C \}$

$D_3 = \{ R_0, R_{120}, R_{240}, M_1, M_2, M_3 \}$

$R_{120} M_1$

$M_1 R_{120} = M_3.$





D_3 is a group.

Chapter-5

Permutation Group

Permutation

$$P_r = \frac{n!}{(n-r)!}$$

Combination

$$^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

How many arrangements of n numbers

$$= {}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \quad (\because 0! = 1)$$

Permutation:

A Permutation of a set A is a function from A to A that is both one-one and onto.

In other words, f is a Permutation of a set A

if (i) $f : A \to A$
(ii) f is one-one
(iii) f is onto

A Permutation group of set A is a set of all permutations of A that forms a group under operation 'composition'.

$$\boxed{\begin{array}{l}\underline{\text{Composition of two functions}} \\ gf(x) = gof(x) = g\{f(x)\}\end{array}}$$

$A = \{1, 2, 3\}$

We define a Permutation α_1 of the set $\{1, 2, 3\}$.

$\alpha_1(1) = 1, \ \alpha_1(2) = 2, \ \alpha_1(3) = 3.$

there are 3! Permutations of A.

$\alpha_2(1) = 2, \ \alpha_2(2) = 1, \ \alpha_2(3) = 3$

$\alpha_3(1) = 1, \ \alpha_3(2) = 3, \ \alpha_3(3) = 2.$

$\alpha_4(1) = 2, \ \alpha_4(2) = 3, \ \alpha_4(3) = 1$

Perm of A.

$\alpha_4(1) = 2, \quad \alpha_4(2) = 3, \quad \alpha_4(3) = 1$
$\alpha_5(1) = 3, \quad \alpha_5(2) = 2, \quad \alpha_5(3) = 1$
$\alpha_6(1) = 3, \quad \alpha_6(2) = 1, \quad \alpha_6(3) = 2.$

the set $G = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$
is a permutation group of a set $A = \{1, 2, 3\}$.

Set of all permutations of set $A = \{1, 2, 3\}$
is denoted by S_3 and is called Symmetric Group
of degree 3.

$$|S_3| = 6 = 3!$$

→ Similarly, S_4 = set of all permutation of
set $A = \{1, 2, 3, 4\}$

and $|S_4| = 4! = 24.$

Symmetric Group:

Let $A = \{1, 2, 3, \dots, n\}$. The set of all
permutations of A is called the Symmetric
Group of degree n and is denoted by S_n.

Elements of S_n is of the form.

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ \alpha(1) & \alpha(2) & \alpha(3) & \cdots & \alpha(n) \end{bmatrix}$$

Clearly, $|S_n| = n!$

Matrix Notation of Permutation

$A = \{1, 2, 3\}$.

$\alpha_2(1) = 2$, $\alpha_2(2) = 1$, $\alpha_2(3) = 3$

$\alpha_3(1) = 1$, $\alpha_3(2) = 3$, $\alpha_3(3) = 2$

$$\alpha_2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

Composition of two Permutations:

$$\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$\alpha\beta(x) = \alpha \circ \beta(x) = \alpha\{\beta(x)\} \quad \boxed{\beta\alpha(x) = \beta\{\alpha(x)\}}$$

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\beta\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$\therefore \quad \alpha\beta \neq \beta\alpha$

$\Rightarrow \quad S_3$ is a non-abelian group.

<u>Note!</u> $S_n \ (n \geq 3)$ is a non-abelian group.

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

$$\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix}, \quad \alpha\beta \neq \beta\alpha.$$

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

$$\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

$$\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

$$\alpha\beta = \beta\alpha.$$

<u>Abelian group</u> $\quad ab = ba \quad \forall\, a, b \in G.$

Hence ... $ab = ba$.

Q. Show that D_4 is a subgroup of S_4. (holds.)

Soln //

$|D_4| = 8$

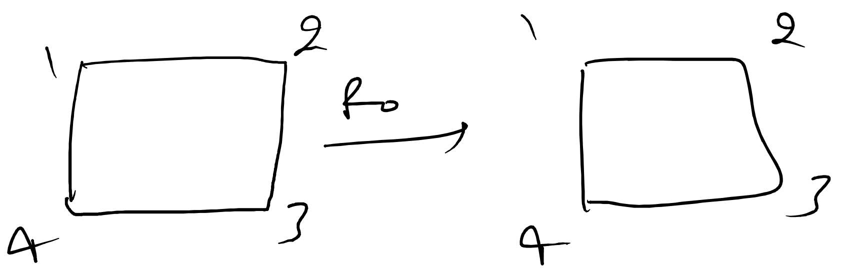
$|S_4| = 24$.

Clearly, $|D_4| < |S_4|$

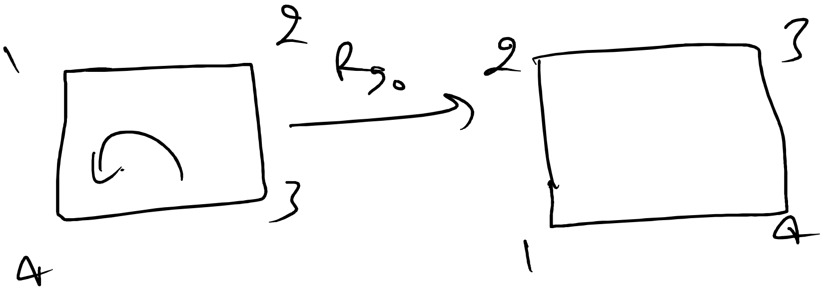
Note: In dihedral group of order $2n$, there are n rotations and n reflections.

Rotations $R = \{R_0, R_{90}, R_{180}, R_{270}\}$

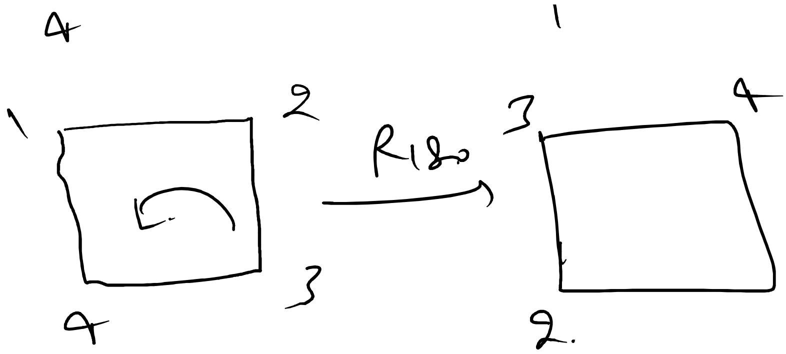
Reflections $F = \{H, V, D, D'\}$.



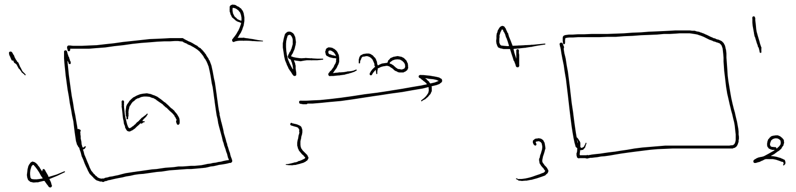
$$\cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \in S_4.$$



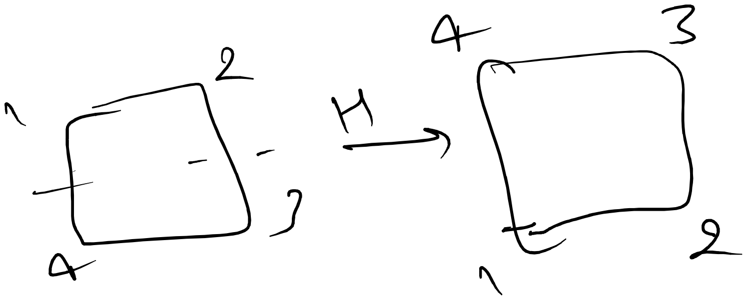
$$\cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \in S_4$$



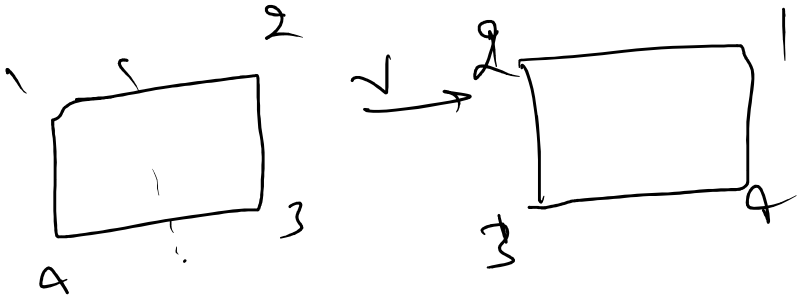
$$\cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \in S_4$$



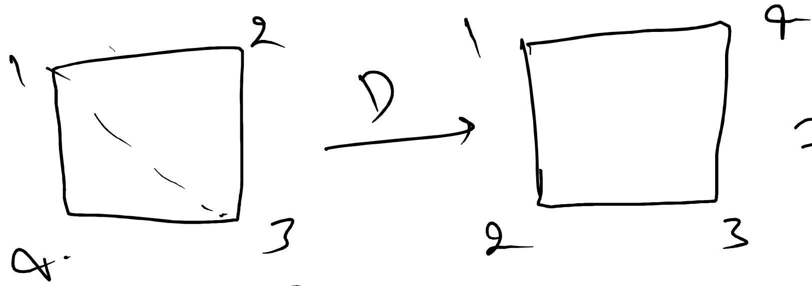
$$\cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \in S_4$$



$$\cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \in S_4$$



$$\cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \in S_4$$



$$\cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \in S_4$$

$$\cong \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \in S_4$$

$\therefore$ elements of D_4 are elements of S_4.

$\therefore$ $|D_4| < |S_4|$

$\Rightarrow$ D_4 is a proper subset of S_4.

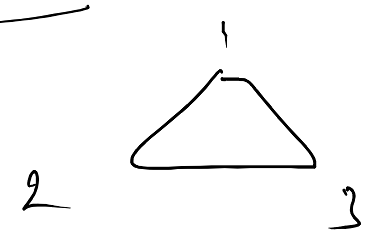
$\because$ D_4 is a group under composition.

$\Rightarrow$ D_4 is a subgroup of S_4.

$\because$ $|D_4| \neq |S_4|$ & $|D_4| = 8 \neq !$

$\Rightarrow$ D_4 is a proper subgroup of S_4.

Q. $\underline{S_3 = D_3}$, $|S_3| = 6$, $|D_3| = 6$



<u>Cycle Notation</u>

$A = \{1, 2, 3\}$

$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$\alpha = (1)(2\ 3)$ | $\beta = (1\ 2)(3)$

$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \cdot & 1 & 4 & 6 & 5 & 3 \end{pmatrix}$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & & & \\ 2 & 1 & 4 & 6 & 5 & 3 \end{pmatrix}$$

$$\alpha = (1\ 2)(3\ 4\ 6)(5)$$

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 4 & 2 \end{bmatrix}$$

$$\beta = (1\ 5\ 4\ 6\ 2\ 3) \rightarrow \text{cycle of length } 6.$$

Cycle: An expression of the form $(a_1\, a_2 \, - - \, a_m)$ is called a cycle of length m or an m-cycle.

Th^m Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

Proof: Let α be a permutation on a set $A = \{1, 2, 3, \ldots, n\}$

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & --- & n \\ \alpha(1) & \alpha(2) & \alpha(3) & --- & \alpha(n) \end{bmatrix}$$

Let us choose a number a_1 from set A.

and let $\alpha(a_1) = a_2$

$\alpha^2(a_1) = \alpha\{\alpha(a_1)\} = \alpha(a_2) = a_3$

$\alpha^3(a_1) = \alpha\{\alpha^2(a_1)\} = \alpha(a_3) = a_4$

and so on until $\alpha^m(a_1) = a_1$ for some m.

$\therefore$ the sequence $a_1, \alpha(a_1), \alpha^2(a_1), \ldots$ is a subset of set $A = \{1, 2, 3, \ldots, n\}$

$\Rightarrow$ the sequence $a_1, \alpha(a_1), \alpha^2(a_1), \ldots$ must be finite.

$\Rightarrow$ there exist some m such that $\alpha^m(a_1) = a_1$

$\therefore$ sequence $a_1, \alpha(a_1), \alpha^2(a_1), \ldots$ is finite

therefore, there must be some repetition

say $\alpha^i(a_1) = \alpha^j(a_1)$ for some $i < j$

$i = 2,\ j = 4.$

$\alpha^2(a_1) = \alpha^4(a_1)$. $\alpha^2(a_1) = a_1$

$\langle (-1)^n \rangle$

$= \{-1, 1, -1, 1, -1, 1, \ldots\}$

$x^2 = 1$

$x^4 = 1$

$\Rightarrow \alpha^{-i}\alpha^j(a_1) = \alpha^{-i}\alpha^i(a_1)$

$\Rightarrow \alpha^{j-i}(a_1) = a_1$

$\Rightarrow m = j - i.$

$= \{-1, 1, -1, 1, -1, 1, \dots\}$ $\Rightarrow$ $m = j - i$.

$= \{-1, 1\}$

$\langle g^m \rangle$ We express this relation

among $a_1, a_2 \dots a_m$ as a cycle.

in permutation α.

$$\alpha = (a_1\ a_2\ a_3 \dots a_m) \dots$$

The three dots indicate the possiblity that

α may not contains all elements of A.

Now, we choose any element b_1 of A, that

not appearing in the first cycle.

Now let $\alpha(b_1) = b_2$.

$$\alpha^2(b_1) = \alpha\{\alpha(b_1)\} = \alpha(b_2) = b_3$$

and so on until $\alpha^r(b_1) = b_1$ for some r.

$\therefore$ it forms a cycle $(b_1\ b_2 \dots b_r)$

This new cycle will have no elements

in common with previously constructed

cycle.

because, if $\alpha^i(a_1) = \alpha^j(b_1)$ for some i and j

$$\Rightarrow \alpha^{-j}\alpha^{i}(a_1) = \alpha^{-j}\alpha^{j}(b_1)$$

$\Rightarrow \quad \alpha^{i-j}(a_1) = b_1$

$\Rightarrow \quad b_1 = a_k$ for some k.

which contradicts the fact that b_1 is not an element of cycle $(a_1\, a_2 \dots a_m)$.

$\therefore$ We may write

$\alpha = (a_1\, a_2 \dots a_m)(b_1 b_2 \dots b_l) \dots$

Continuing this process until we run out of all element of A, our permutation will appear as

$\alpha = (a_1\, a_2 \dots a_m)(b_1\, b_2 \dots b_l) \dots (c_1\, c_2 \dots c_k)$

$\therefore$ every permutation can be written as a cycle or product of disjoint cycles.