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Chapt 10 Q. 41 (Second Isomorphism Theorem)

If K is a subgroup of G and N is a normal subgroup of G, then $\frac{K}{K \cap N}$ is isomorphic to $\frac{KN}{N}$

Sol^n

First isomorphism th^m
If $\phi : G \to \bar{G}$ is homomorphism, then $\frac{G}{\ker \phi} \cong \phi(G)$

If ϕ is onto, then $\frac{G}{\ker \phi} \cong \bar{G}$

TS $\frac{K}{K \cap N} \cong \frac{KN}{N}$ $\quad K \to \frac{KN}{N}$

First we need to show that N is normal subgroup of KN

Let $n \in N$ and $g \in KN$

$\Rightarrow n \in N$ and $g \in G$ $\quad (\because KN \subseteq G)$

$\Rightarrow gng^{-1} \in N$ $\quad (\because N \trianglelefteq G)$

$\Rightarrow N$ is normal subgroup of KN

$\Rightarrow \frac{KN}{N}$ is well defined set.

Now, Define $\phi : K \to \frac{KN}{N}$ such that

$$\phi(k) = kN \quad \forall\, k \in K$$

(i) ϕ is well defined

$k_1 = k_2 \Rightarrow k_1 N = k_2 N$

$\Rightarrow \phi(k_1) = \phi(k_2)$

(ii) ϕ is O.P.

$\phi(k_1 k_2) = k_1 k_2 N = (k_1 N)(k_2 N)$

$= \phi(k_1)\,\phi(k_2)$

$\therefore$ ϕ is homomorphism

Now for every $kN \in \frac{KN}{N}$, $\exists\, k \in K$ such that $\phi(k) = kN$

$G/N \ni \cancel{k}N$

$\therefore$ ϕ is onto $\Rightarrow \phi(K) = \frac{KN}{N}$ — (1)

$$\ker \phi = \{k \in K \mid \phi(k) = N\}$$

$$= \{k \in K \mid kN = N\}$$

$$= \{k \in K \mid k \in N\}$$

$$= K \cap N$$

From first isomorphism thm

k . .

$$\overline{}$$
$$K \cap N \cong \phi(K)$$

$$\Rightarrow \boxed{\frac{K}{K \cap N} \cong \frac{KN}{N}}$$ (using eqn ①)