SECTION - I

Binary operation:

let A be non-empty set. An operation $*$ is said to be binary operation A if

$a * b \in A \ \forall \ a \in A, b \in A$

$\left[\begin{array}{l} \text{i.e } * : A \times A \to A \\ \quad (a, b) \text{ then } a * b \xrightarrow{\text{element}} \in A \end{array} \right]$

example '+' is a binary operation N, W, Z, Q, R

'·' " " " " N, W, Z, Q, R

Subtraction is NOT a binary operation on N

since '−' : $N \times N$ operation

e.g $(1, 1) \xrightarrow{\text{operation}} 1 \underset{\text{operation}}{-} 1 = 0 \notin N$

'−' is binary operation on Z, Q, R

Group: let G be a non-empty set then $(G, *)$ is s.t.b group if the following conditions are satisfied:

1. $a * b \in G \ \forall \ a, b \in G$ (Closure law)
2. $(a * b) * c = a * (b * c) \ \forall \ a, b, c \in G$ (Associative law)
3. There exists an element $e \in G$ such that $a * e = e * a = a \ \forall \ a \in G$ (e is called identity of G)
4. For each $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$ (b is called inverse of a)

Abelian group: A group $(G, *)$ is called an abelian group (or commutative group) if $a * b = b * a$ $\forall \ a, b \in G$

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$(N, +)$ binary operation

example $\mathbb{N}$ (set of natural number) is a NOT a group (because identity $\notin N$)
[there is no element in N, w.r.t '+' we get same element]

① $m+n \in \mathbb{N} \quad \forall m, n \in N$

② $(m+n)+p = m+(n+p) \quad \forall m, n, p \in N$

But there is no element in N

s.t $m + (\text{element}) = m$

(No identity)

② W (set of whole numbers) $(W, +)$

① $m+n \in W \quad \forall m, n \in W$

② $(m+n)+p = m+(n+p) \quad \forall m, n, p \in W$

③ There exists $0 \in W$ s.t

$m+0 = 0+m = m \quad \forall m \in W$

④ But No Inverse

$[\ 1+(-1) = 0$ but $-1 \notin W\ ]$

③ $(Z, +)$ Yes abelian grp

① Same above $m+n \in Z$

④ For each $m \in Z$, there exist $-m \in Z$ such that

$m+(-m) = (-m)+m = 0$

abelian Also $m+n = n+m \quad \forall m, n \in Z$

④ $(Q, +)$ Yes abelian grp

① $\frac{p}{q} + \frac{r}{s} \in Q \quad \forall \frac{p}{q}, \frac{r}{s} \in Q$ where $q, s \neq 0$

② $\left(\frac{p}{q} + \frac{r}{s}\right) + \frac{t}{u}$

$= \frac{p}{q} + \left(\frac{r}{s} + \frac{t}{u}\right)$

$\forall \frac{p}{q}, \frac{r}{s}, \frac{t}{u} \in Q \quad u \neq 0$

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③ There exist $0 = \frac{0}{1} \in \mathbb{Q}$ such that

$$\frac{p}{q} + 0 = 0 + \frac{p}{q} = \frac{p}{q} \quad \forall \frac{p}{q} \in \mathbb{Q}$$

④ For each $\frac{p}{q} \in \mathbb{Q}$ there $-\frac{p}{q} \in \mathbb{Q}$ s.t

$$\frac{p}{q} + \left(-\frac{p}{q}\right) = \left(-\frac{p}{q}\right) + \frac{p}{q} = 0$$

⑤ abelian group $\frac{p}{q} + \frac{r}{s} = \frac{r}{s} + \frac{p}{q} \quad \forall \frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$

$(\mathbb{Q}, +)$ is abelian group

⑤ $(\mathbb{R}, +)$ is also abelian group

⑥ $(\mathbb{Q}^*, \cdot)$ $\mathbb{Q}^* = \mathbb{Q} - \{0\}$

same as ④ e.g grp

① $\frac{p}{q} \cdot \frac{r}{s} \in \mathbb{Q}^* \quad \forall \frac{p}{q}, \frac{r}{s} \in \mathbb{Q}^*$

② $\left(\frac{p}{q} \cdot \frac{r}{s}\right) \cdot \frac{t}{u} = \frac{p}{q} \cdot \left(\frac{r}{s} \cdot \frac{t}{u}\right) \quad \forall \frac{p}{q}, \frac{r}{s}, \frac{t}{u} \in \mathbb{Q}^*$

③ $\exists \; 1 = \frac{1}{1} \in \mathbb{Q}^*$ such that

$$1 \cdot \frac{p}{q} = \frac{p}{q} \cdot 1 = \frac{p}{q} \quad \forall \frac{p}{q} \in \mathbb{Q}^*$$

④ $\forall \; 0 \neq \frac{p}{q} \in \mathbb{Q}^*$ there exist $\frac{q}{p} \in \mathbb{Q}^*$

such that $\frac{p}{q} \cdot \frac{q}{p} = \frac{q}{p} \cdot \frac{p}{q} = 1$

⑤ abelian grp $\frac{p}{q} \cdot \frac{r}{s} = \frac{r}{s} \cdot \frac{p}{q} \quad \forall \frac{p}{q}, \frac{r}{s} \in \mathbb{Q}^*$

(P) $(\mathbb{Q}, \cdot)$ is Not a group

Because $0 \in \mathbb{Q}$ $\frac{0}{1} \in \mathbb{Q}$

But $\nexists \; \frac{p}{q} \in \mathbb{Q}$ such that $0 \cdot \frac{p}{q} \neq \underset{\text{identity}}{1}$

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(7) $(R, \cdot)$ is NOT group
But $(R^*, \cdot)$ is abelian grp $R^* = R - \{0\}$

(8) Matrix $(M_2, +)$ is abelian group
$$M_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2} ; a, b, c, d \in R \right\}$$

(1) $A + B \in M_2 \ \forall \ A, B \in M_2$
(2) $(A+B)+C = A+(B+C) \ \forall \ A, B, C \in M_2$
(3) There exists $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_2$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$$

(4) $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$ there exists $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \in M_2$

such that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$
$$= \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(5) $A + B = B + A \ \forall \ A, B \in M_2$
A is matrix
B is matrix.

$(M_2, +)$ is abelian group