or

$$2a_2 + 6a_3x + a_1x - a_0x + \sum_{n=2}^{\infty} [n(n-1)c_n + (n+2)(n+1)c_{n+2} + nc_n - c_{n-1}]x^n = 0 \quad \text{--- (6)}$$

eqn. (6) is an identity, equating the coefficients of various power of x to zero (we get

$\boxed{a_2 = 0}$ --- (7) $\quad (6a_3 + a_1 - a_0)x = 0$

$$6a_3 + a_1 - a_0 = 0$$

$$a_3 = (a_0 - a_1)/6 \quad \text{--- (8)}$$

& $n(n-1)a_n + (n+2)(n+1)a_{n+2} + na_n - a_{n-1} = 0 \quad \forall\, n \geq 2$

$$a_{n+2} = \frac{a_{n-1} - \cancel{na_n} + \cancel{na_n} - n^2a_n}{(n+1)(n+2)}$$

$$a_{n+2} = \boxed{\frac{a_{n-1} - n^2a_n}{(n+1)(n+2)}} \quad \forall\, n \geq 2 \quad \text{--- (9)}$$

This is called recurrence relation

put $n=2$

$$a_4 = \frac{a_1 - 4a_2}{12} = \frac{a_1}{12} \quad \text{--- (10)} \quad \left(\begin{array}{l}\text{as } c_2 = 0 \\ \text{by (7)}\end{array}\right)$$

put $n=3$

$$a_5 = \frac{a_2 - 9a_3}{20} = \frac{-9}{20}(a_0 - a_1)\cdot\frac{1}{6}$$

putting these values of a's in (3) we get

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x + (a_0 - a_1)\frac{x^3}{6} + \frac{a_1}{12} x^4 - \frac{3}{40}(a_0 - a_1) x^5 + \cdots$$

$$= a_0\left(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - \cdots\right)$$

which is the required sol^n near $x=0$.

a_0 & a_1 are arbitrary constts.

Note :- we can also write

$$y = \sum_{n=0}^{\infty} a_n x^n \quad -(3)$$

$$y' = \sum_{n=0} n a_n x^{n-1} \quad -(4)$$

$$y'' = \sum_{n=0} n(n-1) a_n x^{n-1} \quad -(5)$$

put in eqn. we get

$$(x^2+1) \sum n(n-1) a_n x^{n-1} + x \sum n a_n x^{n-1} - x \sum a_n x^n = 0$$

$$\sum n(n-1) a_n x^{n+1} + \sum n(n-1) a_n x^{n-1} + \sum n a_n x^n - \sum a_n x^{n+1} = 0$$

put $n+1 = n$, i.e. $n = n-1$; $n-1 = n$, $n = n+1$

$$(n-1)(n-2) a_{n-1} - a_{n-1} + (n+1) n\, a_{n+1} - a_n = 0$$

$$[(n-1)(n-2) - 1] a_{n-1} + (n+1) n\, a_{n+1} = a_n$$

$$a_{n+1} = \frac{a_n + [1 - (n-1)(n-2)] a_{n-1}}{n(n+1)}$$

(19) # $(1-x^2)y'' - xy' + 4y = 0$ —— (A)

Sol.ⁿ Dividing by $1-x^2$, $y'' - \frac{x}{1-x^2} y' + \frac{4}{1-x^2} y = 0$ —— (1)

Compare with $y'' + P(x)y' + Q(x)y = 0$ —— (2)

$P = -\frac{x}{1-x^2} = 0 \neq \infty$ at $x=0$,

$Q = \frac{4}{1-x^2} = 4 \neq \infty$ at $x=0$,

P & Q are analytic at $x=0$, so $x=0$ is an ordinary point.

Let the solⁿ be of the form $y = a_0 + a_1x + a_2x^2 + \dots a_kx^k = \sum_{k=0}^{\infty} a_k x^k$ —— (3)

$y' = \sum_k a_k k \cdot x^{k-1}$ —— (4)

$y'' = \sum a_k k(k-1) x^{k-2}$ —— (5)

putting in (A)

$(1-x^2) \sum a_k k(k-1) x^{k-2} - x \sum a_k k x^{k-1} + 4 \sum a_k x^k = 0$

$\sum a_k k(k-1) x^{k-2} - \sum a_k k(k-1) x^k - \sum a_k k x^k + 4 \sum a_k x^k = 0$

$\sum a_k k(k-1) x^{k-2} + \sum [4a_k - a_k \cdot k - a_k \cdot k \cdot (k-1)] x^k = 0$

$\sum a_{k+2}(k+2)(k+1) x^k + \sum [\underbrace{4a_k - a_k k - a_k k \cdot (k-1)}_{= -(k^2-4)}] x^k$

equating the coff of x^k equal to zero

$a_{k+2}(k+2)(k+1) - a_k(k^2-4) = 0$

$a_{k+2}(k+2)(k+1) = a_k(k+2)(k-2)$

$\boxed{a_{k+2} = \left(\frac{k-2}{k+1}\right) \cdot a_k}$

If

$\underline{k=0}$: $a_2 = -2a_0$

$\underline{k=1}$: $a_3 = -\frac{1}{2} a_1$

$\underline{k=2}$: $a_4 = 0$

$\underline{k=3}$: $a_5 = \frac{1}{4} a_3 = \frac{1}{4}(-\frac{1}{2} a_0) = -\frac{1}{8} a_0$

$\underline{k=4}$: $a_6 = \frac{2}{5} \times a_4 = 0$

- - -

putting these into (3)

$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$

$= a_0 + a_1x + (-2a_0)x^2 - \frac{1}{2} a_1 x^3 + 0 - \frac{1}{8} a_0 x^5 + 0 - \dots$

$= a_0(1 - 2x^2) + a_1 x \left(1 - \frac{x^2}{2} - \frac{x^4}{8} + \dots\right)$

Ans.

Frobenius Method

This method is named after a German mathematician F. G. Frobenius (1849–1917). His contribution was in mathematics but specially in the theory of matrices & groups.

This method is employed to find the powerseries solⁿ. of the diff. eqn.

$$y'' + P(x)y' + Q(x)y = 0 \qquad \text{—— (I)}$$

When $x = 0$ is the regular singularity.

& power series solⁿ. will be of the form

Procedure:

- $y = x^m(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$ —— (II)

$$\boxed{y = \sum_{k=0}^{\infty} a_k x^{m+k}}$$ where m is real or complex no.

- Substituting the value of y, y' & y'' in (I)
- find the indicial eqn. (or a quadratic eqn.) by equating to zero the coeffs. of the lowest power of x.
- find roots m_1 & m_2 of the indicial eqn.
- find the values of $a_1, a_2, a_3 \ldots$ in terms of a_0 by equating to zero the coeffs. of other power of x.
- The complete solⁿ. depends on the nature of roots of the indicial eqn.

F-2

Cases :

(I) When roots m_1 & m_2 are distinct & do not differ by an integer
(e.g. $m_1 = 1$, $m_2 = \frac{5}{2}$)

$$y_1 = x^{m_1}(a_0 + a_1 x + a_2 x^2 + \cdots)$$
$$y_2 = x^{m_2}(a_0 + a_2 x + a_2 x^2 + \cdots)$$

Complete solⁿ is given by $\boxed{y = c_1 y_1 + c_2 y_2}$

(II) When roots are equal i.e $m_1 = m_2$.

then $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=m_1}$

∴ Complete solⁿ is given by $\boxed{y = c_1 y_1 + c_2 \left(\frac{\partial y_1}{\partial m}\right)_{m_1}}$

(III) When m_1 & m_2 are distinct & differ by an integer.
(e.g $m_1 = \frac{1}{2}$ & $m_2 = \frac{9}{2}$ or $m_1 = 0$, $m_2 = 4$)

In some cases $\boxed{m_1 < m_2}$.
If some of coeffs. of y series becomes ∞ when $m = m_1$, we modif the form of y replacing a_0 by $b(m - m_1)$. Then the complete solⁿ is given by

$$\boxed{y = c_1 y_{m_2} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_2}} \quad \text{or } c_1 y_2 + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_1}$$

coeff. becomes ∞ when $m = m_2$

~~(IV) Roots are distinct + differing by an integer, making some coeffs. indeterminate~~

Otherwise

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

$m_1 \neq m_2$ & not differ by integer

(I) Solve $4xy'' + 2y' + y = 0$; ($x=0$ is a regular singular point)

Sol^n: Given $4xy'' + 2y' + y = 0$ --- (A)

Dividing by $4x$, we get

$$y'' + \frac{2}{4x}y' + \frac{1}{4x}\cdot y = 0 \quad \text{--- (1)}$$

Compare with $y'' + P(x)y' + Q(x)y = 0$ --- (2)

we get $xP(x) = \frac{2}{4} = \frac{1}{2} \neq \infty$ at $x=0$

& $(x-0)^2 Q(x) = 0 \neq 0$ at $x=0$

so $xP(x)$ & $(x-0)^2 Q(x)$ are analytic at $x=0$.

So $x=0$ is a regular singular point.

Let the sol^n be of the form.

$$y = [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots] x^m = \sum a_k x^{m+k} \quad \text{--- (3)}$$

$$y' = \sum a_k (m+k) x^{m+k-1} \quad \text{--- (4)}$$

$$y'' = \sum a_k (m+k)(m+k-1) x^{m+k-2} \quad \text{--- (5)}$$

putting (3) (4) & (5) in (A) we get

$$4x \sum a_k (m+k)(m+k-1) x^{m+k-2} + 2\sum a_k (m+k) x^{m+k-1} + \sum a_k x^{m+k} = 0$$

$$\Rightarrow 4\sum a_k (m+k)(m+k-1) x^{m+k-1} + 2\sum a_k (m+k) x^{m+k-1} + \sum a_k x^{m+k} = 0$$

$$\Rightarrow \sum [4a_k (m+k)(m+k-1) + 2a_k (m+k)] x^{m+k-1} + \sum a_k x^{m+k} = 0 \quad \text{--- (6)}$$

The coeff. of the lowest degree term x^{m-1} in the identity (6) can be obtained by putting $k=0$ in the 1st summation & equating it to zero. The indicial eq^n is.

$$4a_0 (m)(m-1) + 2a_0 (m) = 0$$

$$2a_0 m[1 + 2(m-1)] = 0$$

ie $m = 0$ & $m = \frac{1}{2}$

E-4

The coeff. of next lowest degree term x^{m} in the identity (6) is obtained by putting $k=1$ in the 1^{st} summation & $k=0$. in the 2^{nd} summation & equating it to zero

$$4a_1 m(m+1) + 2a_1(m+1) + a_0 = 0$$

$$\cancel{2a_1(m+1)[2(m+1)+1]+a_0=0}$$

$$2a_1(m+1)[2m+1] + a_0 = 0$$

$$\boxed{a_1 = -\frac{a_0}{2(m+1)(2m+1)}}$$

Now equating to zero the coeff. of x^{m+k} in (6) we get

$$4a_{k+1}(m+k+1)(m+k) + 2a_{k+1}(m+k+1) + a_k = 0$$

$$2a_{k+1}(m+k+1)[2(m+k)+1] = -a_k$$

$$\boxed{a_{k+1} = -\frac{a_k}{2(m+k+1)[1+2(m+k)]}}$$ This is recurrence relation.

when

$k=0$, $\quad a_1 = -\dfrac{a_0}{2(m+1)(1+2m)}$

$k=1$, $\quad a_2 = -\dfrac{a_1}{2(m+2)(2m+3)} = +\dfrac{a_0}{2(m+1)(1+2m)\,2(m+2)(2m+3)}$

$k=2$, $\quad a_3 = -\dfrac{a_2}{2(m+3)(2m+5)} = -\dfrac{a_0}{2(m+3)(2m+5)\,2(m+1)(1+2m)\,2(m+2)(2m+3)}$

$\vdots$

For $m = 0$

$$a_1 = -\frac{a_0}{2} = -\frac{a_0}{2!}$$

$$a_2 = \frac{a_0}{2\cdot 1\cdot 4\cdot 3} = \frac{a_0}{4!}$$

$$a_3 = -\frac{a_0}{2\cdot 3\cdot 5\cdot 2\cdot 1\cdot 4\cdot 3} = -\frac{a_0}{6!}$$

$\vdots$

For $m = \frac{1}{2}$

$$a_1 = -\frac{a_0}{2\cdot\frac{3}{2}\cdot 2} = \frac{-a_0}{2\cdot 3} = \frac{-a_0}{3!}$$

$$a_2 = +\frac{a_0}{2\cdot 3\cdot 2\cdot\frac{5}{2}\cdot 4} = \frac{a_0}{5!}$$

$$a_3 = -\frac{a_0}{2\cdot 3\cdot 4\cdot 5\cdot 2\frac{7}{2}\cdot 6} = -\frac{a_0}{7!}$$

$\vdots$

~~Hence the complete sol^n is given by~~ $\cancel{y = c_1 y_{m=0} + c_2 y_{m=\frac{1}{2}}}$

$\cancel{y = c_1 y_1 + c_2 y_2}$

Thus for $m=0$, $y_{m=0} = y_1 = x^m(a_0 + a_1 x + a_2 x^2 + \cdots)\Big|_{m=0}$

$$= a_0\left[1 - \frac{1}{2!}x + \frac{1}{4!}x^2 + \cdots\right]$$

$$= a_0\left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} + \cdots\right]$$

$$= a_0 \cos\sqrt{x}$$

Likewise for $m=\frac{1}{2}$, $y_{m=\frac{1}{2}} = y_2 = x^m(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)$

$$y_2 = x^{\frac{1}{2}} a_0\left[1 - \frac{x}{3!} + \frac{x^2}{5!} + \cdots\right]$$

$$= a_0\left[\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} + \cdots\right]$$

$$= a_0 \sin\sqrt{x}$$

Hence complete sol^n is $y = c_1 y_1 + c_2 y_2 = c_1 a_0 \cos\sqrt{x} + c_2 a_0 \sin\sqrt{x}$

$$= c_1' \cos\sqrt{x} + c_2' \sin\sqrt{x} \quad \text{Ans.}$$

(4)

Ex 1: Det. whether $x=0$ is an ordinary point or regular singular point of the diff. eqn. $2x^2\frac{d^2y}{dx^2}+7x(x+1)\frac{dy}{dx}-3y=0$

Sol^n: Dividing the eqn by $2x^2$, we get

$$\frac{d^2y}{dx^2}+\frac{7}{2}\left(\frac{x+1}{x}\right)\frac{dy}{dx}-\frac{3}{2x^2}y=0 \quad ---(1)$$

Comparing (1) with standard eqn $y''+P(x)y'+Qy=0$ --(2)

we have : $P(x)=\frac{7}{2}\left(\frac{x+1}{x}\right)$ and $Q(x)=-\frac{3}{2x^2}$

At $x=0$:

$P(x)=\infty$ and $Q(x)=\infty$ ie not defined.

So, P & Q are not analytic.

$\therefore$ $x=0$ is not an ordinary point.

So $x=0$ is a singular point.

Now $(x-0)P(x)=\frac{7}{2}(x+1)=\frac{7}{2}\neq\infty$ at $x=0$

& $(x-0)^2Q(x)=-\frac{3}{2}\neq\infty$ at $x=0$

So at $x=0$ $(x-0)P(x)$ & $(x-0)^2Q(x)$ are analytic

$\therefore$ $x=0$ is a regular singular point.

EX 2: For diff. eqn. $(x^2-1)y'' + xy' - y = 0$;

$x=0$ is an ordinary point but $x=1$ is regular singular point.

Sol^n: - Dividing by (x^2-1) we get

$$y'' + \frac{x}{x^2-1}y' - \frac{1}{x^2-1}\cdot y = 0 \quad --- (1)$$

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$ --- (2)

we have

$$P(x) = \frac{x}{x^2-1} \quad \text{and} \quad Q(x) = -\frac{1}{x^2-1}$$

At $x=0$:

$P(x) = 0$ and $Q(x) = 1$ ie they are analytic

so $x=0$ is an ordinary point.

At $x=1$: $P(x) = \infty$ and $Q(x) = \infty$. So P & Q are not analytic. $\therefore x=1$ is not an ordinary point but it is a singular point.

Now $(x-1)P(x) = \dfrac{x}{x+1} = \dfrac{1}{2} \neq \infty$ at $x=1$

$(x-1)^2 Q(x) = -1 \neq \infty$ at $x=1$

so both $(x-1)P$ and $(x-1)^2 Q$ are analytic at $x=1$

$\therefore x=1$ is a regular singular point.

Power Series : Problems

Prob 1:- Find the power series solⁿ. of the eqn. $(x^2+1)y''+xy'-xy=0$ in power of x (i.e about $x=0$)

Solⁿ: Dividing above eqn by x^2+1, we get

$$y'' + \frac{x}{x^2+1}y' - \frac{x}{x^2+1}y = 0 \quad --- (1)$$

comparing (1) with $y''+Py'+Qy=0 \quad --- (2)$

we get $P(x) = \frac{x}{x^2+1} = 0$ at $x=0$ ($x^2+1 \neq 0$) and $Q(x) = -\frac{x}{x^2+1} = 0$ at $x=0$ ($x^2+1 \neq 0$)

so $P(x)$ & $Q(x)$ are analytic at $x=0$.

∴ $x=0$ is an ordinary point.

Let $y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{n=0}^{\infty} a_nx^n \quad --- (3)$ be the required solⁿ.

differentating (3) w.r.t. 'x',

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad --- (4)$$

$$\& \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad --- (5)$$

substitude (3), (4) & (5) original eqn we get

$$\Rightarrow (x^2+1)\sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + x\sum_{n=1}^{\infty} na_nx^{n-1} - x\sum_{n=0}^{\infty} a_nx^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_nx^n + \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2} + \sum_{n=1}^{\infty} na_nx^n - \sum_{n=0}^{\infty} a_nx^{n+1} = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1)a_nx^n + \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} na_nx^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$