Periodic function : consider a function $f(x)$ of a variable x. $f(x)$ is called a periodic function of x if it repeats itself regularly over a given interval of x.
(x may be time variable or space variable).

mathematically, A function $f(x)$ is said to be periodic with period T if for all x $\boxed{f(x+T) = f(x)}$ --- (1)

where T is a +ve constt. called period of $f(x)$.

The least value of $T>0$ is called the Least period or simply period of $f(x)$.

If 'n' is any integer then from eqn (1) it is clear that

$$\boxed{f(x+nT) = f(x)} \quad \text{--- (2) for all } 'x'$$

Hence nT is $2T$, $3T$, $4T$, --- etc are also period of $f(x)$.

For example :

$\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ having a period of 2π.

$\tan x$, $\cot x$ having a period of π.

Since $\sin x$ has periods of 2π, 4π, 6π etc ---

so 2π is the Least period.

$\therefore \sin x = \sin(x+2\pi) = \sin(x+4\pi) = \cdots\cdots$ with period 2π.

Similarly $\sin 3x = \sin(3x+2\pi) = \sin 3\left(x+\frac{2\pi}{3}\right)$, with period $\frac{2\pi}{3}$

$\cos 3x = \cos(3x+2\pi) = \cos 3\left(x+\frac{2\pi}{3}\right)$, " " $\frac{2\pi}{3}$.

$\vdots$

So the period of $\left.\begin{matrix}\sin nx\\ \cos nx\end{matrix}\right\}$ is $\boxed{\frac{2\pi}{n}}$

Q:- What is the period of $\sin 7x$ & $\cos 5x$

Sol:- The period of $\sin 7x = \frac{2\pi}{7}$.

The period of $\cos 5x = \frac{2\pi}{5}$.

We also have $\tan 2x = \tan(2x+\pi) = \tan 2\left(x+\frac{\pi}{2}\right)$

So $\tan 2x$ having the period of $\frac{\pi}{2}$

Or in general $\left.\begin{matrix}\tan nx\\ \cot nx\end{matrix}\right] \to$ having period of $\boxed{\frac{\pi}{n}}$.

Q: What is the period of $\tan 7x$, $\cot 5x$.

Sol:- The period of $\tan 7x = \frac{\pi}{7}$.

The period of $\cot 5x = \frac{\pi}{5}$.

$f(x+2T) = f[(x+T)+T] = f(x+T) = f(x)$

& for any integer n

$f(x+nT) = f(x) \ \forall x$.

Orthogonality of sine & cosine functions

$$\boxed{\begin{aligned} &\text{I} \quad \int_{-\pi}^{\pi} \sin mx \, \sin nx \, dx = \pi \delta_{mn} \quad \text{for } m, n \neq 0 \\ &\text{II} \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \pi \delta_{mn} \quad \text{for } m, n \neq 0 \\ &\qquad\qquad = 2\pi \quad \text{when } m = n = 0 \\ &\text{III} \quad \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \quad \text{for all values of } m \,\&\, n \end{aligned}}$$

m & n +ve integers

At first take some definite integrals then we'll prove above relations.

(1) $\int_{-\pi}^{\pi} \sin mx \, dx = -\frac{\cos mx}{m}\Big|_{-\pi}^{\pi} = -\frac{1}{m}\left[\cos m\pi - \cos m\pi\right] = 0$

(2) $\int_{-\pi}^{\pi} \cos mx \, dx = \frac{\sin mx}{m}\Big|_{-\pi}^{\pi} = \frac{1}{m}\left[\sin m\pi + \sin m\pi\right] = 0$

(3) $\int_{-\pi}^{\pi} \sin^2 mx \, dx = \int_{-\pi}^{\pi} \frac{(1-\cos 2mx)}{2} dx = \frac{1}{2}\left[x - \frac{\sin 2mx}{2m}\right]_{-\pi}^{\pi}$

$= \frac{1}{2}\left[\pi - 0 - (-\pi) - 0\right] = \frac{2\pi}{2} = \pi$

(4) $\int_{-\pi}^{\pi} \cos^2 mx \, dx = \int_{-\pi}^{\pi} \frac{(1+\cos 2mx)}{2} dx = \frac{1}{2}\left[x + \frac{\sin 2mx}{2}\right]_{-\pi}^{\pi}$

$= \frac{1}{2}\left[\pi + 0 - (-\pi) + 0\right] = \frac{2\pi}{2} = \pi$

$$\boxed{\begin{aligned} \sin n\pi &= 0 \\ \cos n\pi &= (-1)^n \end{aligned}}$$

(I) $I = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx$

(i) when $m=0$ or $n=0$ or both m & $n=0$

then $\boxed{I=0}$

(ii) when $m \neq n \neq 0$

$$I = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x] \, dx$$

$$I = \frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi}$$

$$\boxed{I = 0}$$

(iii) when $m=n \neq 0$

$$I = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi \quad \therefore \boxed{I=\pi}$$

(iv) ~~when~~ so

$$\boxed{\begin{aligned}\int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \pi \delta_{mn} \\ &= 0 \text{ for } m=0 \text{ or } n=0\end{aligned}} \quad m,n \neq 0$$

(II) $I = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx$

(i) when $m=0$ or $n=0$

then $I = \int_{-\pi}^{\pi} \cos mx \, dx = 0$

$\therefore \boxed{I = 0}$

(ii) when both m & $n = 0$

then $\boxed{I = \int_{-\pi}^{\pi} dx = 2\pi}$

(iii) when $m = n \neq 0$, then

$$I = \int_{-\pi}^{\pi} \cos^2 mx \, dx = \pi$$

$\boxed{I = \pi}$

$$\therefore \boxed{\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \pi \delta_{mn} \; ; \text{ for } m, n \neq 0 \\ &= 2\pi \text{ when both } m \,\&\, n = 0 \end{aligned}}$$

(iv) when $m \neq n, \neq 0$

$$I = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \left(\cos(m+n)x - \cos(m-n)x \right) dx$$

$$I = \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} - \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi}$$

$\boxed{I = 0}$

(II) $I = \int_{-\pi}^{\pi} \sin mx \cos nx \, dx$

(i) when $m=0$, $I = 0$

(ii) when $n=0$ $\quad I = \int_{-\pi}^{\pi} \sin mx \, dx = 0$

(iii) when $m=n$

$$I = \int_{-\pi}^{\pi} \frac{1}{2}\left(2 \sin mx \cos mx\right) dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mx \, dx = 0$$

$$\boxed{I = 0}$$

(iv) when $m \neq n \neq 0$

$$I = \int_{-\pi}^{\pi} \sin mx \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin(m+n)x + \sin(m-n)x \, dx$$

$$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

$$= 0$$

$\therefore$ $\boxed{\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \text{ for all values of } m \,\&\, n}$

Summary for integrals uses in fourier series

$$\boxed{\int_{-\pi}^{\pi} \sin mx\,dx = 0 = \int_{-\pi}^{\pi} \cos mx\,dx}$$

$$\boxed{\begin{aligned} \sin n\pi &= 0 \\ \cos n\pi &= (-1)^n \end{aligned}}$$

$$\boxed{\int_{-\pi}^{\pi} \sin^2 mx\,dx = \pi = \int_{-\pi}^{\pi} \cos^2 mx\,dx}$$

(I) $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \pi\,\delta_{mn}$ for $m,n \neq 0$ → m & n are +ve integers

(II) $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \pi\,\delta_{mn}$ for $m,n \neq 0$ →

$= 2\pi$ for $m \& n = 0$

(III) $\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$ for all values of m & n

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

n = integral

$f(n\pi \pm \theta) = \pm$ same function e.g. $\sin(2\pi \pm \theta) = \pm \sin\theta$

$f(\frac{n\pi}{2} \pm \theta) = \pm$ co-function e.g. $\cos(\frac{3\pi}{2} \pm \theta) = \pm \sin\theta$

OR

$f(n\cdot\frac{\pi}{2} \pm \theta) = \pm$ same function when n = even

$= \pm$ co-function when n = odd

e.g. $\sin(4620) = \sin[90^\circ(52) - 60^\circ] = \sin(60^\circ) = -\sqrt{3}/2$

$\operatorname{cosec}(270^\circ - \theta) = \operatorname{cosec}(90^\circ \times (3) - \theta) = -\sec\theta$

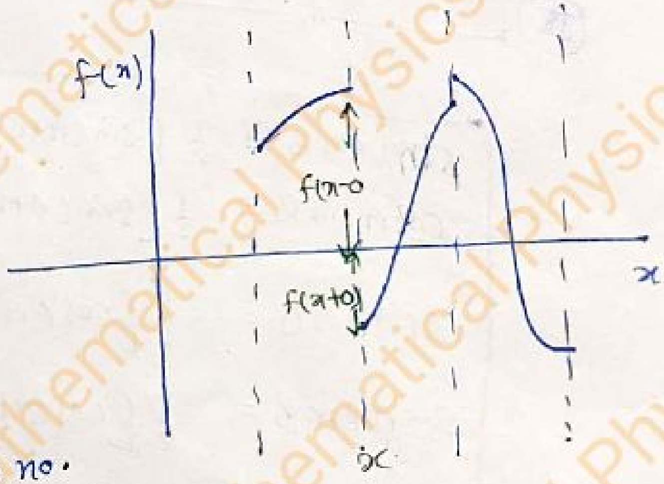
Jean-Baptist Joseph Fourier (1768-1830), French

physicist & mathematician lived & taught in Paris.
In 1822 he published fourier series.
So fourier series after his name

He developed the theory of heat conduction (heat eqns.)

Peter Dirichlet was German mathematician (1805-1859)

meaning of piecewise continuous function



A function f(x) is said to be piecewise continuous in an interval if

(i) f(x) has a finite no. of maxima & minima

(ii) f(x) has a finite no. of discontinuities.

OR

(i) the interval can be divided into a finite no. of subintervals in each of which f(x) is continuous.

(ii) the limits of f(x) as x approaches the endpoints of each subinterval are finite.

Fourier Series

J. B Fourier in 1822 published a very useful theorem by which any complex, periodic function can be analysed.

If $f(x)$ is a singlevalued periodic function with period 2π + integrable over a period & is a piecewise continuous function then $f(x)$ may be expressed as a series of sines & cosines (i.e. $f(x)$ can be represented by a trignometric series) in the form

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)} \quad \text{--- (1)}$$

eqn. (1) is called fourier series expansion of $f(x)$.

where a_0, a_n, b_n are called the fourier coefficients.

eqn (1) having a period of 2π in interval $(-\pi, \pi)$

Determination of fourier coefficients (Euler's Formulae):

(i) To find a_0: Integrating eqn (1) b/w the limits $-\pi$ to π on b.s. w.r.t

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\, dx$$

$$= a_0 \cdot 2\pi + 0 + 0$$

$$\therefore \boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx}$$

Note: of course any interval $c \le x \le c+2\pi$

i.e. $(c, c+2\pi)$

(ii). <u>To find a_n</u> : Multiplying ① by $\cos mx$, where m is a fixed +ve integer & integrate b/w the limits $-\pi$ to π

$$\therefore \int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_0\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos mx\cos nx\,dx + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\cos mx\sin nx\,dx$$

$$= a_0\cdot 0 + a_n\pi + 0$$

$$\therefore \boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx} \qquad n = 1, 2, 3 \ldots$$

(iii) <u>To find b_n</u> :- Multiplying ① by $\sin nx$, & integrate b/w the limits $-\pi$ to π

$$\therefore \int_{-\pi}^{\pi} f(x)\sin nx\,dx = a_0\int_{-\pi}^{\pi}\sin nx\,dx + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nx\sin nx\,dx + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin^2 nx\,dx$$

$$= a_0\cdot 0 + a_n\cdot 0 + b_n\pi$$

$$\therefore \boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx} \qquad n = 1, 2, 3 \ldots$$

So we have the fourier series

$$f(x) = \frac{a_0}{} + \sum_{n\geq 1}^{\infty} a_n \cos nx + \sum_{n\geq 1}^{\infty} b_n \sin nx \quad \text{——— (A)}$$

where
$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad n=1,2,3\ldots \quad (B) \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad n=1,2,3\ldots \quad (C) \end{aligned}}$$

The Fourier series expansion of a function is possible provided it satisfies a set of conditions known as Dirichlet Conditions.

Dirichlet Conditions : Suppose a function $f(x)$ for the interval $(-\pi, \pi)$ then.

1) $f(x)$. must be single valued & well defined.

2) $f(x)$ & $f'(x)$ must be continuous or piecewise continuous i.e.
   $f(x)$ has only a finite no. of maxima & minima;
   $f(x)$ has only a finite no. of discontinuities. + Note on next page

3). $f(x)$ must satisfy the periodic condition
   $f(x+2\pi) = f(x)$ for all values of x outside $[-\pi, \pi]$

These conditions are sufficient but not necessary. If the conditions are satisfied then the convergence is Guaranteed.

If a function $f(x)$ is well defined & bounded in interval $-\pi < x < \pi$ & has only a finite no. of maxima & minima and has only a finite no. of discontinuities also for other values of x it satisfies the condition of periodicity $f(x+2\pi) = f(x)$ then the function $f(x)$ may be expanded in Fourier series which converges to $\frac{1}{2}[f(x-0) + f(x+0)]$ at every value of x [i.e. hence it converges to $f(x)$ at point where $f(x)$ is continuous]

Note:-

If $f(x)$ & $f'(x)$ are piecewise continuous in $[-\pi, \pi]$

then series (A) with coefficients (B) & (C) converges to

(i) $f(x)$ if x is a point of continuity

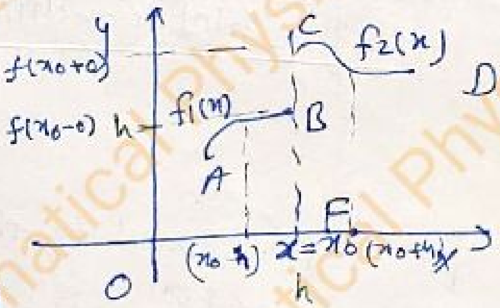
(ii) $\dfrac{f(x+0)+f(x-0)}{2}$ if x is a point of discontinuity

Fourier series for Discontinuous function :

Let the function $f(x)$ be defined by $f(x) = f_1(x), \quad c < x < x_0$

$\qquad\qquad\qquad\qquad = f_2(x), \quad x_0 < x < c+2\pi$

where x_0 is the point of discontinuity in interval $(c, c+2\pi)$.



so finding fourier coefficients

$$a_0 = \frac{1}{2\pi}\left[\int_c^{x_0} f_1(x)\,dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

If $x = x_0$ is the point of finite discontinuity the the sum of fourier series = mean of limits on right & left.

$$\boxed{\begin{aligned} f(x=x_0) &= \frac{1}{2}\left[\lim_{h\to 0} f(x_0-h) + \lim_{h\to 0} f(x_0+h)\right] \\ &= \frac{1}{2}\left[f(x_0-0)+f(x_0+0)\right] \quad \text{—— (I)} \\ &= \frac{1}{2}(FB+FC) \end{aligned}}$$

so (I) converges to $\boxed{\frac{1}{2}[f(x-0)+f(x+0)]}$

Note: at a point of discontinuity $x = c$

$$\boxed{f(x) = \frac{1}{2}\left[f(c-0)+f(c+0)\right]}$$

$$\boxed{\int_{-L}^{L} \sin\frac{n\pi x}{L}\,dx = \int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx = 0}$$ if $n = 1, 2, 3 \cdots$

Orthogonality of sine & Cosine functions

(I) $$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = L\,\delta_{mn}$$

and $n = 1, 2, 3 \cdots$

(II) $$\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \quad \forall\ m \ \& \ n$$

$\Rightarrow$ The result (I) & (II) remains valid when the limit of integration $-L, L$ are replaced by $c, c+2L$

$\Rightarrow$ At a point of discontinuity, fourier series gives the value of $f(x)$ as the arithmetic mean of left & right limits

At the point of discontinuity at $x = x_0$

$$\boxed{f(x = x_0) = \frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right]}$$

Useful formulae / Integrals

$$\int_0^{2\pi} \sin nx\, dx = \int_0^{2\pi} \cos nx\, dx = 0$$

$$\int_0^{2\pi} \sin^2 nx\, dx = \int_0^{2\pi} \cos^2 nx\, dx = \pi$$

$\int_0^{2\pi} \sin mx \sin nx\, dx$ $= \pi$ for $m \neq n$, $m \,\&\, n = 1, 2, 3, \dots$

$= 0$ for $m = 0$

$\int_0^{2\pi} \cos mx \cos nx\, dx$ $= \pi$ for $m \neq n$

$= 2\pi$ when $m \,\&\, n = 0$

$$\int_0^{2\pi} \sin mx \cos nx\, dx = 0 \quad \forall\ m \,\&\, n$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

Que Represent the function $f(x) = x$, $-\pi < x < \pi$ in the form of a fourier series & deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$

Sol: The fourier series expansion of the function $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin x \quad \text{---} (1)$$

Where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi} - \int_0^{\pi} 1\cdot\left(-\frac{\cos nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi(-\cos n\pi)}{n} + \left(\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{\cancel{\pi}}{n}(-\cos n\pi) + 0 + 0\right]$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$\therefore\ x = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} + \frac{2}{n}(-1)^{n+1}\sin nx$ put $x = \frac{\pi}{2}$

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots}$$