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Q. If H is cyclic, then $\phi(H)$ is cyclic.

~~Proof:~~ If $H = \langle a \rangle$, then $\phi(H) = \langle \phi(a) \rangle$.

Given that H is cyclic subgroup of G.

then, there exist $a \in H$ such that

$$H = \langle a \rangle \quad \text{———} \ (1)$$

Given that $\phi : G \to \bar{G}$ is a group homomorphism.

Let e & $\bar{e}$ be the identity elements of G & $\bar{G}$ respectively.

Now, $\phi(H) = \{ \phi(h) \mid h \in H \}$ ——— (2)

We claim that $\phi(H) = \langle \phi(a) \rangle$.

Let $x \in \phi(H)$ be an arbitrary element.

$\because x \in \phi(H) \Rightarrow \exists\, h \in H$ such that

$$\phi(h) = x \quad \text{———} \ (3)$$

$\because h \in H$ & $H = \langle a \rangle$

$\therefore h = a^k$ for some $k \in \mathbb{Z}$.

From eqn (3), $\phi(a^k) = x$

$\therefore (\phi(a))^k$ $[\because \phi$ is a

$\Rightarrow \; (\phi(a))^k = x \quad \left[\because \phi \text{ is a homomorphism}\right]$

$\Rightarrow x \in \langle \phi(a) \rangle$.

$\therefore \; \phi(H) \subseteq \langle \phi(a) \rangle$ ——— (4)

$\because a \in H \Rightarrow \phi(a) \in \phi(H)$

$\Rightarrow \langle \phi(a) \rangle \subseteq \phi(H)$ ——— (5)

From eqn (4) & (5), $\phi(H) = \langle \phi(a) \rangle$

$\therefore \phi(H)$ is cyclic.

Q. Show that $\phi : \mathbb{C}^* \to \mathbb{C}^*$ given by $\phi(x) = x^4$ is a homomorphism. Find all the elements that map to 2.

Sol^n// $\mathbb{C}^* \to$ Set of all complex numbers without zero.

$\mathbb{C}^*$ is a group under the operation 'multiplication'

TS// ϕ is a homomorphism.

(i) ϕ is well defined
let $x, y \in \mathbb{C}^*$
$x = y$
$\Rightarrow x^4 = y^4$

(ii) ϕ is O.P.
$\phi(xy) = (xy)^4$
$= x^4 y^4$

$x = y$

$\Rightarrow x^4 = y^4$

$\Rightarrow \phi(x) = \phi(y)$.

ϕ is well defined

$= x^4 y^4$

$= \phi(x)\,\phi(y)$

ϕ is O.P.

Thus, ϕ is a homomorphism from $\mathbb{C}^*$ to $\mathbb{C}^*$.

$$\{x \in \mathbb{C}^* \mid \phi(x) = 2\} = \phi^{-1}(2)$$

Thm If $\phi(g) = g'$, then $\phi^{-1}(g') = g \operatorname{Ker}\phi$

If $\phi(x) = x^4 = 2 \Rightarrow x^4 = 2 \Rightarrow x = 2^{1/4}$.

$$\operatorname{Ker}\phi = \{x \in \mathbb{C}^* \mid \phi(x) = 1\}$$
$$= \{x \in \mathbb{C}^* \mid x^4 = 1\}$$
$$= \{1, -1, i, -i\}$$

$\phi(2^{1/4}) = 2$

$\Rightarrow g = 2^{1/4},\ g' = 2$

$\phi(g) = g'$

$\phi^{-1}(g') = g \operatorname{Ker}\phi$

$$\therefore \phi^{-1}(2) = 2^{1/4}\operatorname{Ker}\phi = 2^{1/4}\{1, -1, i, -i\}$$
$$= \{2^{1/4}, -2^{1/4}, i2^{1/4}, -i2^{1/4}\}$$

Thus, the set of all elements that map to 2

is $\{2^{1/4}, -2^{1/4}, i2^{1/4}, -i2^{1/4}\}$