Contour Integration

$$\boxed{\begin{aligned} \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} &= \frac{2\pi}{\sqrt{a^2-b^2}} \ ; \quad a>b>0 \\ \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} &= \frac{2\pi}{\sqrt{a^2-b^2}} \ ; \quad a>|b| \end{aligned}}$$

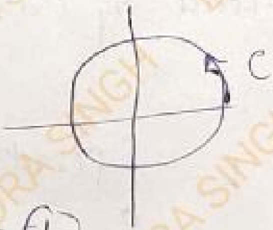
Show that

① $$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Sol^n: Let $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{1}{a+b\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)} d\theta$

put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$$I = \int_C \frac{1}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)} \cdot \frac{dz}{iz}$$ where C denotes the unit circle $|z|=1$



$$I = \frac{2}{ib} \int_C \frac{dz}{\left(z^2+1+\frac{2a}{b}z\right)} = \int_C f(z)\,dz \quad \text{—— ①}$$

poles are given by $z^2 + \frac{2a}{b}z + 1 = 0$ ie $(z-\alpha)(z-\beta) = 0$

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b} \ ; \quad \beta = \frac{-a-\sqrt{a^2-b^2}}{b}$$

we have $\alpha+\beta = -\frac{2a}{b}$

& $\alpha\beta = 1$

as $a > b > 0$

$\therefore |\beta| > 1$, also $|\alpha||\beta| = 1$ ie $|\alpha| < 1$

$\therefore$ α lies inside C and β lies outside contour C

So $z = \alpha$ is a simple pole

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Now residue at $z=\alpha$ is given by

$$\text{Res at } z=\alpha = \lim_{z\to\alpha} (z-\alpha)\cdot\frac{2}{ib(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{2}{ib(z-\beta)} = \frac{2}{ib(\alpha-\beta)}$$

$$= \frac{\cancel{2}}{i\cancel{b}\left(\frac{\cancel{2}\sqrt{a^2-b^2}}{\cancel{b}}\right)}$$

$$= \frac{1}{i\sqrt{a^2-b^2}}$$

Hence by Cauchy's Residue theorem

$$I = \int_C f(z)\,dz = 2\pi i \times [\text{sum of residues at poles w/in } C]$$

$$= 2\pi i \times \frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \boxed{\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}}$$

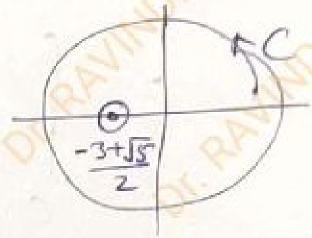
④ Evaluate $\int_0^{\pi} \frac{d\theta}{3+2\cos\theta}$

Sol^n: let $I = \int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\left[\frac{e^{i\theta}+e^{-i\theta}}{2}\right]}$ ---- (A)

put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$\therefore I = \frac{1}{2}\int_C \frac{dz}{iz\left[3+\frac{2}{2}\left(z+\frac{1}{z}\right)\right]}$ where C is a unit circle $|z|=1$

$= \frac{1}{2i}\int_C \frac{dz}{z^2+3z+1} = \int_C f(z)\,dz$ ---- (1)



poles are given by $z^2+3z+1=0$

$z = \frac{-3\pm\sqrt{5}}{2}$

$z = \frac{-3+\sqrt{5}}{2}$; $\frac{-3-\sqrt{5}}{2}$

only $z = \frac{-3+\sqrt{5}}{2}$ lies inside C

So, therefore, residue at $z = \frac{-3+\sqrt{5}}{2}$ is

$= \lim_{z\to\frac{-3+\sqrt{5}}{2}} \cancel{\left[z-\left(\frac{-3+\sqrt{5}}{2}\right)\right]}\cdot\frac{1}{\cancel{\left[z-\left(\frac{-3+\sqrt{5}}{2}\right)\right]}\left\{z+\left(\frac{3+\sqrt{5}}{2}\right)\right\}}\cdot\frac{1}{2i}$

$= \frac{1}{\frac{-3+\sqrt{5}}{2}+\frac{3+\sqrt{5}}{2}}\cdot\frac{1}{2i} = \frac{1}{\sqrt{5}}\cdot\frac{1}{2i}$

So by Cauchy's residue theorem

$I = \int_C f(z)\,dz = 2\pi i\times$[sum of residues at poles w/in C]

$= 2\pi i\times\frac{1}{\sqrt{5}}\frac{1}{2i} = \frac{\pi}{\sqrt{5}}$

$\int_0^{2\pi}\frac{d\theta}{3+2\cos\theta} =$ [illegible] $= \frac{\pi}{\sqrt{5}}$

So $\int_0^{\pi}\frac{d\theta}{3+2\cos\theta} = \frac{1\cdot\pi}{2\sqrt{5}} = \frac{\pi}{2\sqrt{5}}$ Ans [see (A)]

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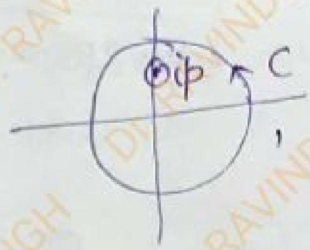
(7) Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2}$ where $p^2<1$

Sol:- Let $\displaystyle I = \int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \int_0^{2\pi} \frac{d\theta}{1-2p\left[\frac{e^{i\theta}-e^{-i\theta}}{2i}\right]+p^2}$

put $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

$$\therefore I = \int_C \frac{1}{\left(1-\frac{2p}{2i}\left[z-\frac{1}{z}\right]+p^2\right)}\cdot\frac{dz}{iz}$$

where C denotes a unit circle $|z|=1$



$$= \int_C \frac{1}{\left[1+ip\left(z-\frac{1}{z}\right)+p^2\right]}\cdot\frac{dz}{iz}$$

$$= \int_C \frac{dz}{zi-pz^2+p+p^2zi}$$

$$= \int_C \frac{dz}{(zi+p)(izp+1)} = \int_C f(z)\,dz \quad \text{--- (1)}$$

as $-pz^2+p^2zi+zi+p$
$= pz(-z+pi)+zi+p$
$= \frac{pz}{(-i)}(zi+p)+zi+p$
$= (zi+p)\left(\frac{pz}{-i}+1\right)$
$= (zi+p)(ipz+1)$

poles are given by
$(zi+p)(izp+1)=0$
ie $z = ip$ & $z = \frac{i}{p}$

only $z = ip$ lies inside contour C.

so residue at $z = ip$ is

$$= \lim_{z\to ip} (z-ip)\cdot\frac{1}{(zi+p)(izp+1)}$$

$$= \lim_{z\to ip} \frac{\cancel{(z-ip)}}{i\cancel{(z-ip)}}\cdot\frac{1}{(izp+1)}$$

$$= \frac{1}{i(1-p^2)}$$

So by Cauchy's residue theorem

$$I = \int_C f(z)dz = 2\pi i \times [\text{Sum of residues at poles w/in } C]$$

$$= 2\pi i \times \frac{1}{i(1-p^2)} = \frac{2\pi}{1-p^2}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \frac{2\pi}{1-p^2}$$ Ans.

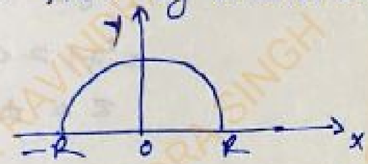
Evaluation of $\int_{-\infty}^{\infty} \frac{f_1(x)}{f_2(x)}\, dx$

where $f_1(x)$ & $f_2(x)$ are polynomial in x.

Such integrals can be reduced to contour integrals, if

(i) $f_2(x)$ has no real roots.

(ii) the degree of $f_2(x)$ is greater than that of $f_1(x)$ by at least two.



Procedure: let $f(x) = \frac{f_1(x)}{f_2(x)}$

Consider $\int_C f(z)\,dz$ where C is a curve consisting of the upper half C_R of circle $|z| = R$ and real axis from $-R$ to R. If there is no pole on real axis, the circle $|z| = R$ which is arbitrary can be taken such that there is no singularity on its circumference C_R in the upper half of the plane, but possibly some poles inside the contour C specified above.

Using Cauchy's residue theorem we have

$$\int_C f(z)\,dz = 2\pi i \times [\text{sum of residues within } C]$$

i.e. $$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\ [\text{sum of residues w/in } C]$$

or $$\int_{-R}^{R} f(x)\,dx = -\int_{C_R} f(z)\,dz + 2\pi i\,[\text{sum of residues w/in } C]$$

$$\therefore \lim_{R\to\infty} \int_{-\infty}^{\infty} f(x)\,dx = -\lim_{R\to\infty} \int_{C_R} f(z)\,dz + 2\pi i\,[\text{sum of residues w/in } C] \quad \text{------ (1)}$$

but $$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = \lim_{R\to\infty} \int_0^{\pi} f(Re^{i\theta}) \cdot Rie^{i\theta}\, d\theta$$ when put $z = Re^{i\theta}$

$= 0$ so by (1) we have

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\ [\text{sum of residues w/in } C]}$$

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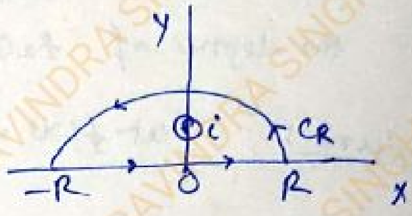
EXAMPLE 1 :- Evaluate $\int_0^\infty \frac{\cos mx}{(x^2+1)}\,dx$

Sol :- Consider the integral $\int_C f(z)\,dz$ where $f(z) = \frac{e^{imz}}{z^2+1}$, taken round the closed contour c consisting of upper half C_R of large semi circle $|z| = R$ and real axis from $-R$ to R

the poles of $f(z)$ are

$z^2+1 = 0$

$z = \pm i$, only $z = i$ lies inside c.



So, Residue at $z = i$

$$\lim_{z\to i} \frac{(z-i)\,e^{imz}}{(z+i)(z-i)} = \lim_{z\to i} \frac{e^{imz}}{(z+i)} = \frac{e^{-m}}{2i}$$

So by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residue}$$

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \times \frac{e^{-m}}{2i}$$

$$\text{or } \int_{-R}^{R} \frac{\cos mx}{x^2+1}\,dx + \int_{C_R} \frac{e^{imz}}{(z^2+1)}\,dz = \pi e^{-m} \quad \text{----}(1)$$

$$\text{Now } \left|\int_{C_R} \frac{e^{imz}}{(z^2+1)}\,dz\right| \le \int_{C_R} \left|\frac{e^{imz}}{z^2+1}\right| |dz|$$

$$\le \int_C \frac{e^{im|z|}}{|z|^2+1} |dz| \le \int_0^\pi \frac{1}{R^2+1} R\,d\theta$$

put $z = Re^{i\theta}$

$|dz| = R\,d\theta$

$$= \frac{R}{R^2+1}\pi \to 0 \text{ as } R\to\infty$$

so ① becomes

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\, dx = \pi e^{-m}$$

$$\boxed{\int_{0}^{\infty} \frac{\cos mx}{x^2+1}\, dx = \frac{\pi}{2} e^{-m}}$$ Ans!

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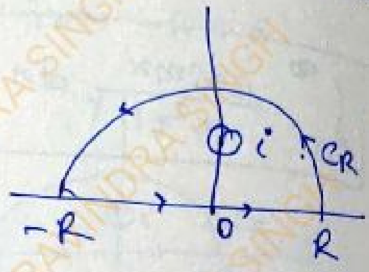
EXAMPLE 2: Evaluate $\int_0^\infty \frac{dx}{(1+x^2)^3}$

Sol:- Consider $\int_C f(z)\,dz$ where $f(z) = \frac{1}{(1+z^2)^3}$, taken round the closed contour C consisting of upper half CR of semicircle $|z|=R$ and real axis from $-R$ to R.



The poles of $f(z)$ are

$$(1+z^2)^3 = 0$$

$$z = \pm i \quad \text{each pole of order 3}$$

only $z=i$ lies inside C.

So residue at $z=i$ is given by

$$= \lim_{z\to i} \frac{1}{2!}\left[\frac{d^2}{dz^2}\frac{\cancel{(z-i)^3}}{(z+i)^3\cancel{(z-i)^3}}\right]$$

$$= \lim_{z\to i} \frac{1}{2}\left[\frac{d^2}{dz^2}\frac{1}{(z+i)^3}\right]$$

$$= \lim_{z\to i} \frac{1}{2}\left[\frac{12}{(z+i)^5}\right] = \frac{1}{2}\frac{12}{(i+i)^5} = \frac{6}{(2i)^5} = \frac{6}{32i}$$

$$= \frac{3}{16i}$$

Hence by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues within C}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{CR} f(z)\,dz = 2\pi i \times \frac{3}{16 i}$$

$$\int_{-R}^{R} \frac{1}{(1+x^2)^3}\,dx + \int_{CR}\frac{dz}{(1+z^2)^3} = \frac{2\pi\times 3}{16} \quad \text{-----(1)}$$

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Now $\left|\int_{C_R} \frac{dz}{(1+z^2)^3}\right| \le \int_{C_R} \left|\frac{1}{(1+z^2)^3}\right| |dz| \le \int_{C_R} \frac{1}{(1+|z|^2)^3} |dz|$

$$= \int_0^{\pi} \frac{R\,d\theta}{R^2+1} \to 0 \text{ as } R \to \infty \quad \text{put } z = Re^{i\theta} \quad \therefore |dz| = R\,d\theta$$

Therefore ① becomes

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^3} = \frac{2\pi \times 3}{16}$$

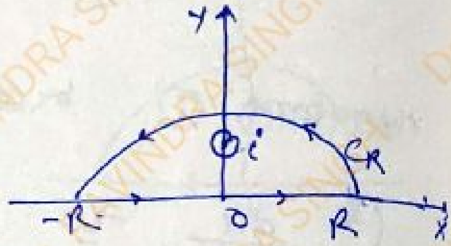
$$2\int_0^{\infty} \frac{dx}{(1+x^2)^3} = \frac{2\pi \times 3}{16}$$

$$\boxed{\int_0^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{16}}$$

Ans.

EXAMPLE 3:- Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^3}\,dx$

Sol:- Consider $\int_C f(z)\,dz$ where $f(z) = \frac{z^2}{(1+z^2)^3}$ taken around the closed contour C consisting of upper half C_R of large semicircle $|z| = R$ and real axis from $-R$ to R



The poles of $f(z)$ are

$(1+z^2)^3 = 0$

$z = \pm i$ each pole of order 3

only $z = i$ lies inside C.

So residue at $z = i$, put $z = i + t$

$$\therefore \frac{z^2}{(1+z^2)^3} = \frac{(i+t)^2}{[1+(i+t)^2]^3} = \frac{-1+t^2+2it}{[1-1+t^2+2it]^3}$$

$$= \frac{(-1+2it+t^2)}{(2it)^3\left[1+\frac{1}{2i}t\right]^3} = \frac{(-1+2it+t^2)}{-8it^3}\left[1+\frac{t}{2i}\right]^{-3}$$

$$= -\frac{1}{8i}\left(-\frac{1}{t^3}+\frac{2i}{t^2}+\frac{1}{t}\right)\left(1-\frac{3t}{2i}+\frac{(-3)(-4)}{2!}\frac{t^2}{-4}+\cdots\right.$$

$$= -\frac{1}{8i}\left(-\frac{1}{t^3}+\frac{2i}{t^2}+\frac{1}{t}\right)\left[1-\frac{3}{2i}t-\frac{3}{2}t^2-\cdots\right]$$

So coefficient of $\frac{1}{t} = -\frac{1}{8i}\left(\frac{3}{2}-3+1\right) = -\frac{i}{16}$

So residue at $z = i$ = coeff. of $\frac{1}{t} = -\frac{i}{16}$

So by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of residues within } C$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \times \frac{-i}{16}$$

$$\text{or } \int_{-R}^{R} \frac{x^2}{(1+x^2)^3}\,dx + \int_{C_R} \frac{z^2}{(1+z^2)^3}\,dz = \frac{2\pi}{16} \quad \text{------} (1)$$

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Now $\left|\int_{C_R} \frac{z^2}{(1+z^2)^3}\,dz\right| \leq \int_{C_R} \left|\frac{z^2}{(1+z^2)^3}\right| |dz| \leq \int_{C_R} \frac{|z|^2}{(1+|z|^2)^3} |dz|$

$\leq \int_{C_R} \frac{|z|^2}{(|z|^2-1)^3} |dz|$

$= \int_0^{2\pi} \frac{R^3}{(R^2-1)^3}\,d\theta$ put $z = Re^{i\theta}$, $|dz| = R\,d\theta$

$= 0$ as $R \to \infty$ {since $|z_1+z_2| \geq |z_1| - |z_2|$

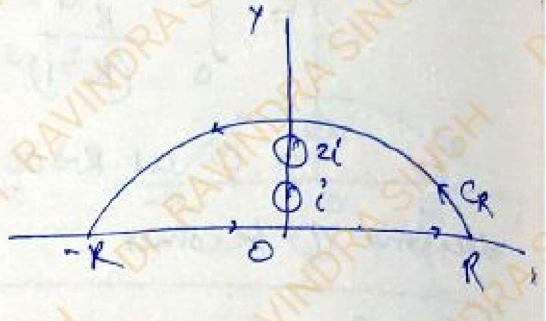
Therefore (1) becomes

$$\boxed{\int \frac{x^2}{(1+x^2)^3}\,dx = \frac{2\pi}{16} = \frac{\pi}{8}}$$

EXAMPLE 4: Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+4)}$

Sol :- Consider $\int_C f(z)dz$, where $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ taken round the closed contour c consisting of upper half C_R of large semi circle $|z|=r$ and real axis from $-R$ to R

The poles of $f(z)$ are



$(z^2+1)(z^2+4) = 0$

$z = \pm i, \; z = \pm 2i$

only $z = i$ & $2i$ lies inside c.

So residue at $z = i$

$$\lim_{z \to i} \frac{\cancel{(z-i)}\, z^2}{(z+i)\cancel{(z-i)}(z+2i)(z-2i)}$$

$$= \lim_{z \to i} \frac{z^2}{(z+i)(z+2i)(z-2i)} = \frac{-1}{6i}$$

residue at $z = 2i$

$$\lim_{z \to 2i} \frac{\cancel{(z-2i)}\, z^2}{(z+i)(z-i)\cancel{(z-2i)}(z+2i)}$$

$$= \lim_{z \to 2i} \frac{z^2}{(z+i)(z-i)(z+2i)} = \frac{1}{3i}$$

So by Cauchy's residue theorem

$$\int_C f(z)dz = 2\pi i \times [\text{sum of residues within } C]$$

$$\therefore \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \times \left[\frac{1}{3i} - \frac{1}{6i}\right]$$

$$\int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)}dx + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)}dz = \frac{2\pi}{6} \quad \text{——— (1)}$$

now

$$\left|\int_{C_R}\frac{z^2}{(z^2+1)(z^2+4)}dz\right| \le \int_{C_R}\left|\frac{z^2}{(z^2+1)(z^2+4)}\right||dz| \le \int_{C_R}\frac{|z|^2}{(|z|^2+1)(|z|^2+4)}|dz|$$

$$\le \int_{C_R}\frac{|z|^2}{(|z^2|-1)(|z|^2-4)}|dz|$$

put $\boxed{z = Re^{i\theta}}$

$|dz| = |Rie^{i\theta}d\theta| = R\,d\theta$

$$\le \int_0^{\pi}\frac{R^2\,R\,d\theta}{(R^2-1)(R^2-4)} \to 0 \text{ as } R\to\infty$$

$\left\{\text{as } |z_1+z_2| \ge |z_1|-|z_2|\right.$

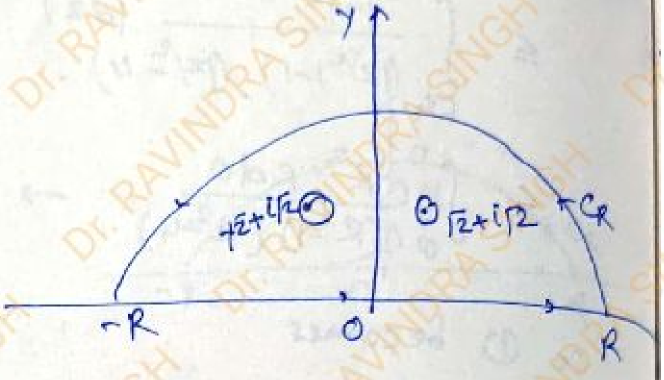
so (1) becomes

$$\boxed{\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3}}$$

EXAMPLE 5: Evaluate $\int_0^\infty \frac{dx}{x^4+16}$

Sol:- Consider $\int_C f(z)dz$, where $f(z) = \frac{1}{z^4+16}$ taken round the closed contour C consisting of upperhalf C_R of semicircle $|z| = R$ and real axis from $-R$ to R.

The poles of $f(z)$ are given by



$$z^4+16 = 0$$

$$z^4 = -16 = 16e^{i(2n+1)\pi}$$

$$z = 2e^{i(2n+1)\pi/4}$$

$$n = 0, 1, 2, 3$$

If $n=0$, $z = 2e^{i\pi/4} = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = 2\left[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = \sqrt{2}+i\sqrt{2}$

$n=1$, $z = 2e^{3i\pi/4} = 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = 2\left[\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right] = -\sqrt{2}+i\sqrt{2}$

$n=2$, $z = 2e^{5i\pi/4} = 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = 2\left[\frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = -\sqrt{2}-i\sqrt{2}$

$n=3$, $z = 2e^{7i\pi/4} = 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = 2\left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right] = \sqrt{2}-i\sqrt{2}$

but only $\sqrt{2}+i\sqrt{2}$ and $-\sqrt{2}+i\sqrt{2}$ lies inside C.

So residue (at $z = 2e^{i\pi/4}$) $= \left(\frac{1}{\frac{d}{dz}[z^4+16]}\right)_{z=2e^{i\pi/4}} = \left(\frac{1}{4z^3}\right)_{z=2e^{i\pi/4}}$

$$= \frac{1}{32e^{3i\pi/4}} = \frac{1}{32}e^{-3i\pi/4}$$

also residue (at $z = 2e^{3i\pi/4}$) $= \left[\frac{1}{\frac{d}{dz}(z^4+16)}\right]_{z=2e^{3i\pi/4}} = \left(\frac{1}{4z^3}\right)_{z=2e^{3i\pi/4}}$

$$= \frac{1}{32e^{9i\pi/4}} = \frac{1}{32}e^{-9i\pi/4}$$

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By Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \times [\text{sum of residues within } C]$$

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \times \left[\frac{1}{32} e^{-3i\pi/4} + \frac{1}{32} e^{-9i\pi/4}\right]$$

$$\int_{-R}^{R} \frac{dx}{x^4+16} + \int_{C_R} \frac{dz}{z^4+16} = \frac{\pi i}{16}\left[e^{-3i\pi/4} + e^{-9i\pi/4}\right] \text{ - - - - - - } (1)$$

Now $\left|\int_{C_R} \frac{dz}{z^4+16}\right| \le \int_{C_R} \left|\frac{1}{z^4+16}\right| |dz| \le \int_{C_R} \frac{1}{|z^4|+16} |dz| \le \int_{C_R} \frac{|dz|}{(|z|^4-16)}$

put $z = Re^{i\theta}$

$$\le \int_0^{\pi} \frac{R\,d\theta}{R^4-16} \to 0 \text{ as } R \to \infty$$

so (1) becomes

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+16} = \frac{\pi i}{16}\left[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i\sin\frac{9\pi}{4}\right]$$

$$= \frac{\pi i}{16}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]$$

$$= \frac{\pi i}{16}\frac{(-2i)}{\sqrt{2}} = \frac{\pi}{8\sqrt{2}} = \frac{\sqrt{2}\,\pi}{16}$$

$$2\int_0^{\infty} \frac{dx}{x^4+16} = \frac{\sqrt{2}\,\pi}{16}$$

$$\boxed{\int_0^{\infty} \frac{dx}{x^4+16} = \frac{\sqrt{2}\,\pi}{32}} \quad \text{or} \quad \boxed{\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\sqrt{2}\,\pi}{4a^3}}$$

This is general result.

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