(2.5)

Lineer Diophantine equation. ①

Equation of the form $ax+by=c$ where $a,b,c \in \mathbb{Z}$ and a, b are not both zero.

where x, y are two unknowns

A pair (x_0, y_0) is called solution if

$$ax_0 + by_0 = c$$

Lineer Diophantine equation can have number of solution

e.g $3x+6y=18$

$(10,-2)$, $(4,1)$, $(-6,6)$ are solutions of $3x+6y=18$ and many more

$3(10)+6(-2)=18$, $3(4)+6(1)=18$, $3(-6)+6(6)=18$

and so on...

② $2x+10y=17$, it has no solution since LHS is an even integer whatever choice of x and y and RHS is odd.

To find, whether eqn has No, solution, infinite solution or unique solution, we have theorem.

Theorem (2.9) The lineer Diophantine equation $ax+by=c$ has solution iff $d|c$ where $d=\gcd(a,b)$.

If x_0, y_0 is any particular solution of this eqn then all other solutions are given by

$$x = x_0 + \left(\frac{b}{d}\right)t \quad \text{and} \quad y = y_0 - \left(\frac{a}{d}\right)t$$

where t is any arbitrary integer.

Proof) Consider $ax+by=c$ has solution say (x_0, y_0)

let $d=\gcd(a,b)$

By defn of GCD, $d|a$ and $d|b$

There exists integers r and s s.t

$a = dr$ and $b = ds$

Now, we have $ax_0 + by_0 = c$

then $c = drx_0 + dsy_0 \Rightarrow c = d(rx_0 + sy_0)$

$c = d(rx_0 + sy_0)$

$\Rightarrow d \mid c$

Conversly, let $d \mid c$

there exist integer say t i.e $c = dt$

By theorem (2.3) Given integers a and b, not both of which are zero, there exist integers x and y s.t

$$d = \gcd(a,b) = ax + by$$

By using thm. (2.3)

integers x_0 and y_0 can be found s.t

$$d = ax_0 + by_0$$

$$\Rightarrow dt = (ax_0 + by_0)t, \ t \in \mathbb{Z}$$

$$\Rightarrow c = ax_0t + by_0t$$

$$\Rightarrow c = a(tx_0) + b(ty_0)$$

Hence, eqn $ax + by = c$ has $x = tx_0$ & $y = ty_0$ is a particular solution.

Second part let us suppose (x_0, y_0) is solution of $ax + by = c$

let (x_1, y_1) be another solution

i.e $ax_0 + by_0 = c = ax_1 + by_1$

$$\Rightarrow a(x_0 - x_1) = -b(y_0 - y_1)$$

Since $\gcd(a,b) = d$, there exists integers r and s s.t $a = dr$ and $b = ds$

we get.

$$dr(x_0 - x_1) = -ds(y_0 - y_1)$$

$$\Rightarrow r(x_0 - x_1) = -s(y_0 - y_1) \Rightarrow r(x_1 - x_0) = s(y_0 - y_1)$$

$$\Rightarrow r \mid s(y_0 - y_1) \quad (\because x_0 - x_1 \in \mathbb{Z})$$

Also $\gcd(r,s) = 1$

Since $a = dr \Rightarrow r = \frac{a}{d}$

and $b = ds \Rightarrow s = \frac{b}{d}$

$d = \gcd(a,b)$

explaination

$$\therefore \gcd(r,s) = \gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$$

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So,

If $\gcd(a,b)=d$

$\Rightarrow \gcd\left(\frac{a}{d}, \frac{b}{d}\right)=1$ (2)

Now, $\gcd(r,s) = \gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$

$\therefore \gcd(r,s)=1$

Now, Euclid's lemma : If $a \mid bc$ and $\gcd(a,b)=1$
$\Rightarrow a \mid c$

$\therefore$ Using Euclid's lemma

$r \mid (y_0 - y_1)$

$\Rightarrow y_0 - y_1 = rt$ for some integer t.

$\Rightarrow y_0 - rt = y_1$

$\Rightarrow y_1 = y_0 - \frac{a}{d} t$ $(\because a = dr)$

Since, we have

$r(x_1 - x_0) = s(y_0 - y_1)$

$r(x_1 - x_0) = srt$

$\Rightarrow x_1 - x_0 = st$

$\Rightarrow x_1 = x_0 + \frac{b}{d} t$ $(\because b = ds)$

then $x_1 = x_0 + \frac{b}{d} t$ and $y_1 = y_0 - \left(\frac{a}{d}\right) t$ where t is other solution of $ax + by = c$

Since, $ax_1 + by_1 = a\left(x_0 + \frac{b}{d} t\right) + b\left(y_0 - \left(\frac{a}{d}\right) t\right)$

$= (ax_0 + by_0) + \left(\frac{ab}{d} - \frac{ab}{d}\right) t$

$= c + 0 \cdot t$

$= c$

thus, there are an infinite number of solutions of given eq^n, one for each value of t.

$t = 0, \pm 1, \pm 2 \ldots$
initial

Corollary: If $\gcd(a,b)=1$ and if x_0, y_0 is a particular solution of linear Diophantine equation $ax+by=c$, then all solution are given by

$$x = x_0 + bt, \quad y = y_0 - at \quad ; t \in \mathbb{Z}$$

Example (2.5) A customer bought a dozen (दर्जन) pieces of fruit, apples & oranges for \$ 1.32. If an apple costs 3 (cents) more than an orange and more apples than oranges were purchased, how many pieces of each kind were bought

(Sol^n) let x be number of apples, y be no. of oranges. and z be the cost (in cents) of an orange.

(dozen) $\therefore \quad x+y=12$

and given cond^n is $(z+3)x + zy = 132$

$\Rightarrow \quad 3x + z(x+y) = 132$ (1 dollar = 100 cents

$\Rightarrow 3x + z(12) = 132$ or 1.32 doller

$\Rightarrow \quad x + 4z = 44$ = 132 cents)

Now $\gcd(1,4)=1$

By theorem (2.9), solution exists.

we have $1 = 1 \times 4 + 1 \times (-3)$ (relation)

As $\gcd(1,4) = 1 \Rightarrow 1 \,\&\, 4$ are relatively prime.

[using thm (2.4) let $a, b \in \mathbb{Z}$ and not both zero, then $(a,b) = 1 \Leftrightarrow$ there exists integers x and y st $1 = ax + by$]

$1 = 1(-3) + 4 \times 1$

multiply by 44

$44 = 1 \times (-132) + 4 \times 44$

$\Rightarrow \quad x_0 = -132, \; z_0 = 44$

Hence $(-132, 44)$ is one of solution

All other solution are

$x_1 = x_0 + bt \Rightarrow x_1 = -132 + 4t$, $t \in \mathbb{Z}$

$y_1 = y_0 - at \Rightarrow y_1 = 44 - t$ for some $t \in \mathbb{Z}$

Not all the choices for t furnish solution to original problem

only values of t that ensure $12 \geq x > 6$ 2 ~~values~~ should be considered.

apples are more than oranges x apples

x y y oranges

$6+6 = 12$ (Not possible

$5+7$ (Not possible

so, $6 < x \leq 12$

i.e $12 \geq -132 + 4t > 6$

$\Rightarrow 12 \geq -132 + 4t$

$\Rightarrow \boxed{t \leq 36}$

and $-132 + 4t > 6 \Rightarrow \boxed{t > 34\frac{1}{2}}$

The only integral values of t satisfy both inequalities are $t = 35$ and $t = 36$

$x = 8 = -132 + 4(35)$

$y = 9 = 44 - 35$

x = number apples

y = number of oranges

$t = 36$

$x = 12 = -132 + 4(36)$

$y = 8 = 44 - 36$

Excercise (Pg 37) Q - 1, 2, 3, 6, 7

(Q1) which of following Diophantine eqn cannot be solved?

(1) $6x + 51y = 22$ here $a = 6$, $b = 51$ $c = 22$

$\gcd(6, 51)$

(long div) $51 = 8 \times 6 + 3$

$6 = 2 \times 3 + 0$

$\therefore d = 3$

$\gcd(6, 51) = 3$ but $3 \nmid 22$

$\therefore$ solⁿ doesn't exist

$33x + 14y = 115$ (4)

$\gcd(33, 14) = 1$

Now $1 \mid 115$

∴ soln exists

$33 = 14 \times 2 + 5$

$14 = 5 \times 2 + 4$

$5 = 4 \times 1 + 1$ gcd

$4 = 1 \times 4 + 0$

(c) $14x + 35y = 93$

$\gcd(14, 35) = 7$

$7 \nmid 93$

No soln

$35 = 14 \times 2 + 7$

$14 = 7 \times 2 + 0$

∴ $d = 7$

(Q2) Determine all soln in integers of following D-E.

(a) $56x + 72y = 40$

$\gcd(56, 72)$

$72 = 1 \times 56 + 16$

$56 = 3 \times 16 + 8$

$16 = 2 \times 8 + 0$

$\gcd(56, 72) = 8$

Since $8 \mid 40$

then soln exists

$8 = 56 - 16 \times 3$

$= 56 - (72 - 1 \times 56) \times 3$

$8 = 4 \times 56 - 72 \times 3$

multiply by 5

$40 = 20 \times 56 + (-15) \times 72$

thus $x_0 = 20$, $y_0 = -15$ is one of soln

All other soln

$x_1 = x_0 + \left(\frac{b}{d}\right) t_1$, $y = y_0 - \left(\frac{a}{d}\right) t_1$; $t \in \mathbb{Z}$

$x_1 = 20 + 9t$, $y = -15 - 7t$; $t \in \mathbb{Z}$

then $(x_0, y_0) = (18, -3)$ sol^n ①

other sol^n $x_1 = x_0 + \left(\frac{b}{d}\right)t$, $y_1 = y_0 - \left(\frac{a}{d}\right)t$

$x_1 = 18 + 23t$, $y = -3 - 4t$; $t \in \mathbb{Z}$

(Q-3) Determine all sol^n in positive integers of D.E

① $18x + 5y = 48$

$\gcd(18, 5) = 1$

$1 \mid 48$ sol^n exists

$18 = 5 \times 3 + 3$

$5 = 3 \times 1 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

$$1 = 3 - 2$$
$$= 3 - (5 - 3)$$
$$= 2 \times 3 - 5$$
$$= 2(18 - 5 \times 3) - 5$$
$$= 2 \times 18 + (-7) \times 5$$

multiply by 48

$$48 = 96 \times 18 + (-336) \times 5$$

$x_0 = 96$, $y_0 = -336$

$x_1 = 96 + 5t$, $y_1 = -336 - 18t$; $t \in \mathbb{Z}$

to obtain positive sol^n $x_1 > 0$, $y_1 > 0$

$96 + 5t > 0$, $-336 - 18t > 0$

$5t > -96$, $-18t > 336$

$t > -96/5$, $t < -\frac{336}{18}$

$$-\frac{96}{5} < t < -\frac{336}{18}$$

$$-19.2 < t < -18.5$$

$$t = -19$$

$x = 96 + 5(-19)$, $y = -336 - 18(-19)$

$x = 1$, $y = 6$

only positive sol^n

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