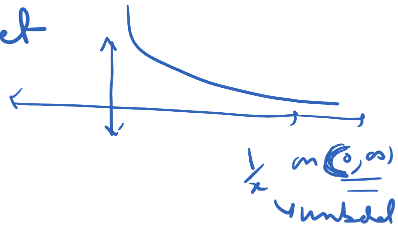
5th October 2020

Boundedness theorem: Continuous function on closed & bdd interval is bdd therein

f is bdd on A if $\exists\ M>0 : |f(x)| \le M\ \forall x \in A$

i.e. $f(A) = \{f(x) \mid x \in A\}$ is a bdd set

Range set



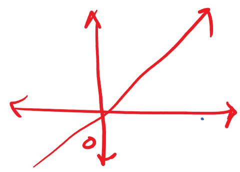
→ cts f^n
→ closed interval
→ bdd interval

Note: Each hypothesis of boundedness theorem is mandatory.

Example 5.3.3

① Interval must be bounded

Let $f : [0,\infty) \to \mathbb{R}$ be def as $f(x) = x$ $\quad I = [0,\infty)$



① I is closed

② f is cts on I

③ I is NOT bounded

we see that f is not bounded (which implies that bddness thm is not longer true)

Let $M \in \mathbb{R}^+$ let $x = M + \frac{1}{2}$ then $x \in [0,\infty) = I$

and $f(x) = x > M$

i.e for any $M>0$, $\exists\ x \in I : f(x) > M \Rightarrow f$ is unbdd on I

Q// State boundedness theorem. Show that the cond^n/hypothesis that I is bdd can not be relaxed.

② Interval must be closed

Let $f : (0,1] \to \mathbb{R}$ be def as $f(x) = \frac{1}{x}\ \forall x \in (0,1]$

$I = (0,1]$

① I is bdd

② f is cts on I

③ I is not closed interval

we see that f is unbdd (i.e. bdd thm fails)

Let $M>0$. let $x = \frac{1}{M+1}$ then $x \in I$ $\left(M>0 \Rightarrow M+1>0 \Rightarrow \frac{1}{M+1} > 0 \right)$

Let $M > 0$. let $x = \frac{1}{M+1}$ then $x \in I = [0,1]$

$\left(\begin{array}{l} M>0 \Rightarrow M+1>0 \Rightarrow \frac{1}{M+1} > 0 \\ M>0 \Rightarrow M+1 > 1 \Rightarrow \frac{1}{M+1} < 1 \\ 0 < \frac{1}{M+1} < 1 \end{array} \right)$

& $f(x) = \frac{1}{x} = M+1 > M$

$\therefore \forall M > 0 \ \exists x \in I : f(x) > M$

$\Rightarrow f$ is unbdd

③ function Must be continuous

Let $I = [0,1]$ & $f: I \to \mathbb{R}$ be def as

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Then ① I is closed

② I is bdd

③ f is discts at $0 \in I$ ($\because \lim_{x \to 0} \frac{1}{x}$ does not exist)

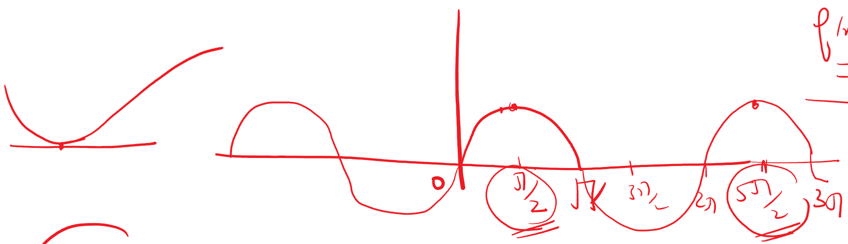
we see that f is unbdd.

f is unbdd

Absolute Maximum of a function

$U \in A$

$f(U) \geq f(x)$



$f(\pi/2) \geq f(x) \quad \forall x \in [0, \pi]$

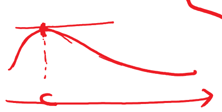
point of abs max.

Def. Let $A \subseteq \mathbb{R}$ & $f: A \to \mathbb{R}$, we say that f has an absolute maximum (minima) on A if $\exists U \in A : f(U) \geq f(x) \ \forall x \in A$

Ex

'U' is called point of abs maxima of f on A

$f'(c) = 0$



eg. ① $f(x) = \sin x$ on $[0, \pi]$

$\pi/2$ is a pt of abs max as $f(\pi/2) = 1 \geq f(x)$
$\forall x \in [0,\pi]$

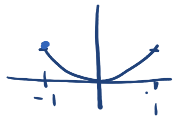
② $f(x) = \sin x$ on $[0, 3\pi] = I$

$\pi/2, 5\pi/2$ are points of abs maxima on I

$\because f(\pi/2) = f(5\pi/2) \geq f(x) \quad \forall x \in I$

Notes: ① We note from eg ② that point of abs max may not be unique

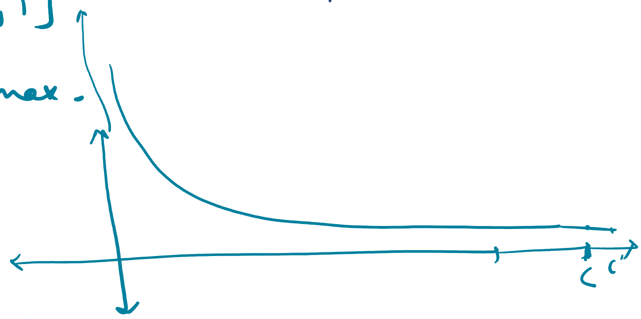
another eg $f(x) = x^2$ on $[-1,1]$



$f(\pm 1) = 1 \geq f(x) \quad \forall x \in [-1,1]$

$1, -1$ are points of abs. max.

② $f(x) = \frac{1}{x}, \quad x \in (0, \infty)$



As f is unbdd above

$\therefore \nexists\, u \in (0,\infty) : f(u) \geq f(x)$

$\Rightarrow f$ DOES NOT HAVE abs Max on $(0,\infty)$

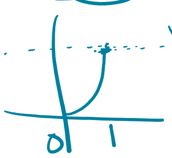
$f(x) > 0$

$\inf f(x) = \{\frac{1}{x} \mid x \in (0,\infty)\} = 0$

$\frac{1}{u} \neq 0$ for any $u > 0$

$\nexists\, u \in (0,\infty) : f(u) = 0$

$\Rightarrow f$ Does not have abs min on $(0,\infty)$



$f(x) \leq M$

$M - \varepsilon$

$M = 1$

$[0,1)$

Max f Sup f ?   Min f Inf f ?

$f(u)$

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