11.2 Limit and Continuity

$(x,y) \in \mathbb{R}^2$ s.t $\sqrt{(x-a)^2+(y-b)^2} < r$, $r>0$.

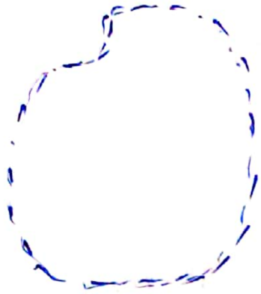


Fig 1 open set

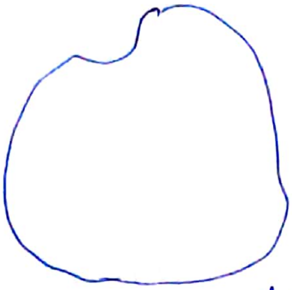
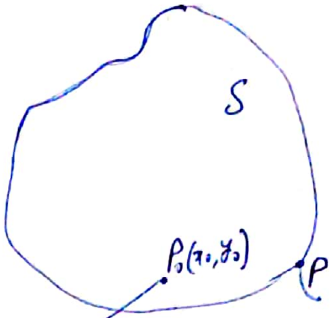


Fig 2 closed set



interior pt Fig 3 boundary pt

(2) **Open Disc** (x,y,z) s.t $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2} < r$, $r>0$.

(3) **Defn** $\lim\limits_{x\to c} f(x) = L$ if $\forall\ \varepsilon>0\ \exists\ \delta>0$ s.t

whenever $0<|x-c|<\delta$, $|f(x)-L|<\varepsilon$

(4) **Defn** $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$, if $\forall\ \varepsilon>0\ \exists\ \delta>0$ s.t

whenever $(x,y)\in D$ and $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$

then $|f(x,y)-L|<\varepsilon$

(5) Ex $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+x-xy-y}{x-y} = \lim\limits_{(x,y)\to(0,0)} \dfrac{(x+1)(x-y)}{x-y} = \lim\limits_{(x,y)\to(0,0)} x+1$ (for $x\neq y$)

$= 1$

(6) ST $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$ does not exist

(7) $y=0$, $f(x,0) = \dfrac{0}{x^2} = 0$ $\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$ along x axis

(8) $x=0$, $f(0,y) = \dfrac{0}{y^2} = 0$ $\therefore \lim f(x,y) = 0$ along Y axis

(9) $x=y$ $f(x,x) = \dfrac{2x^2}{2x^2} = 1$ $\therefore f(x,y)\to 1$ along $x=y$

(10) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist

(11) $y=0$, $f(x,0) = \frac{0}{x^4} = 0$ $f(x,y) \to 0$

(12) $x=0$, $f(0,y) = \frac{0}{y^2} = 0$, $f(x,y) \to 0$

(13) $x=y$, $f(x,x) = \frac{x^3}{x^4+x^2} = \frac{x}{1+x^2} \to 0$ as $x \to 0$

(14) $y = x^2$, $f(x,x^2) = \frac{x^4}{2x^4} = \frac{1}{2}$ $\therefore f(x,y) \to \frac{1}{2}$ as $(x,y)\to(0,0)$

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(15) Rule $\lim\limits_{(x,y)\to(x_0,y_0)} (f+g)(x,y) = \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) + \lim\limits_{(x,y)\to(x_0,y_0)} g(x,y)$

(16) Ex $\lim\limits_{(x,y)\to(3,-4)} x^2+xy+y^2 = 3^2 + (3)(-4) + (-4)^2 = 13$

(17) Ex $\lim\limits_{(x,y)\to(1,2)} \frac{2xy}{x^2+y^2} = \frac{\lim\limits_{(x,y)\to(1,2)} 2xy}{\lim\limits_{(x,y)\to(1,2)} x^2+y^2} = \frac{2\times1\times2}{1^2+2^2} = \frac{4}{5}$

(18) Exercise 11.2 (12) $\lim\limits_{(x,y)\to(1,2)} \frac{(x^2-1)(y^2-4)}{(x-1)(y-2)} = \lim\limits_{(x,y)\to(1,2)} (x+1)(y+2) = 2\times4$

(13) $\lim\limits_{(x,y)\to(0,0)} \frac{e^x \tan^{-1} y}{y} = \lim\limits_{x\to0} e^x \cdot \lim\limits_{y\to0} \frac{\tan^{-1}y}{y} = 1.1 = 1$

(15) $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}$, (14) $\lim \frac{x^2y^2}{x^2+y^2}$

(28) $\lim\limits_{(x,y)\to(0,0)} \left[1 - \frac{\sin(x^2+y^2)}{x^2+y^2}\right] = \lim\limits_{r\to0} \left(1 - \frac{\sin r^2}{r^2}\right)$

(30) $f(x,y) = \frac{x^4y^4}{(x^2+y^4)^3}$ (31) $f(x,y) = \frac{x-y^2}{x^2+y^2}$

(32) $\frac{x^2+y}{x^2+y^2}$ (33) $\frac{x^2y^2}{x^4+y^4}$

Rules for limit of a function of 2 variables.

Suppose $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = M$. Then

1) $\lim\limits_{(x,y)\to(x_0,y_0)} [af](x,y) = af$ (for const a)

2) $\lim\limits_{(x,y)\to(x_0,y_0)} (f+g)(x,y) = L+M$

3) $\lim\limits_{(x,y)\to(x_0,y_0)} [fg](x,y) = LM$

4) $\lim\limits_{(x,y)\to(x_0,y_0)} \left[\frac{f}{g}\right](x,y) = \frac{L}{M}$ if $M \neq 0$.

Continuity

The function $f(x,y)$ is continuous at the pt (x_0,y_0) if

1) $f(x_0,y_0)$ is defined

2) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exist

3) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

* The function f is cts on a set S if it is cts at each pt in S.

* Using basic properties of limits we can show that if f and g are both cts on set S then $(f+g)$, af, fg and f/g $(g \neq 0)$ and $\sqrt[n]{f}$ (wherever defined) are also cts on S.

* If F is a fn of 2 variables at cts at (x_0, y_0) and G is a fn of one variable and cts at $F(x_0, y_0)$ Then $G \circ F$ is cts at (x_0, y_0).

* A polynomial fn of 2 variables is cts throughout x,y plane.

* A rational function of 2 variables is cts wherever the denominator poly. is not zero.

Example 5 Test for continuity:

(a) $f(x,y) = \dfrac{x-y}{x^2+y^2}$

being rational fn it is cts everywhere except where the denominator fn is 0 i.e $(0,0)$.

(b) $f(x,y) = \dfrac{1}{y-x^2}$

cts everywhere except at $y-x^2=0$ i.e except on pts lying on parabola $y=x^2$.

<u>Example 6</u> Show that $f(x,y) = \begin{cases} y \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

is cts at $(0,0)$.

<u>Soln.</u> We need to show that for any $\varepsilon > 0$ $\exists$ $\delta > 0$

s.t

$|f(x,y) - f(0,0)| < \varepsilon$, whenever $0 < \sqrt{x^2+y^2} < \delta$
or $0 < x^2+y^2 < \delta^2$

ie $\left|y \sin \frac{1}{x}\right| < \varepsilon$ whenever $0 < x^2+y^2 < \delta^2$

$\left|y \sin \frac{1}{x}\right| \leq |y|$ $\because \left|\sin \frac{1}{x}\right| \leq 1$

$\therefore$ whenever $0 < x^2+y^2 < \delta^2$ then $y^2 < \delta^2$

and $\left|y \sin \frac{1}{x}\right| < \delta$

$\therefore$ if we take $\varepsilon = \delta$ Then

$|f(x,y) - f(0,0)| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$

<u>Limit and Continuity for functions of 3 variables</u>

$\lim_{(x,y,z) \to (x_0,y_0,z_0)} f(x,y,z) = L$ if for any $\varepsilon > 0$

$\exists$ $\delta > 0$ s.t $|f(x,y,z) - L| < \varepsilon$ whenever

$0 < \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} < \delta$.

The function $f(x,y,z)$ is continuous at $P(x_0,y_0,z_0)$ if

1) $f(x_0,y_0,z_0)$ is defined

2) $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z)$ exists

3) $\lim\limits_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$

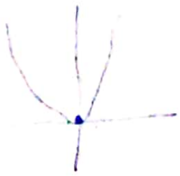
<u>Example 7</u> for what pts (x,y,z), the function

$$f(x,y,z) = \frac{z}{\sqrt{x^2+y^2-2z}}$$ is discts.

Function is cts except where it is not defined i.e

for $x^2+y^2-2z \leq 0$

i.e inside or on the paraboloid $z = \frac{1}{2}(x^2+y^2)$. the function will not be cts.



(12) $\lim\limits_{(x,y)\to(1,2)} \dfrac{(x^2-1)(y^2-4)}{(x-1)(y-2)} = \lim\limits_{(x,y)\to(1,2)} (x+1)(y+2) = 2\times 4 = 8$

(13) $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^x \tan^{-1} y}{y} = \lim\limits_{x\to 0} e^x . \lim\limits_{y\to 0} \dfrac{\tan^{-1} y}{y} = 1 . \dfrac{\lim\limits_{y\to 0} \frac{1}{1+y^2}}{\lim\limits_{y\to 0} 1} = 1$

(14) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2.0}{x^2+0}$

$= 0$

$\because$ if limit exists it is same along every dir
$\therefore$ we take along x axis

(15) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x+y)}{x+y}$. Put $x+y=Z$

$= \lim\limits_{Z\to 0} \dfrac{\sin Z}{Z} = 1$

(27) $\lim\limits_{(x,y)\to(0,0)} (\sin x - \cos y) = \lim\limits_{x\to 0} \sin x - \lim\limits_{y\to 0} \cos y = 0-1=-1$

(25) $\lim\limits_{(x,y)\to(0,0)} \left[1 - \dfrac{\sin(x^2+y^2)}{x^2+y^2}\right]$ Put $x^2+y^2=Z$

$= \lim\limits_{Z\to 0} \left[1 - \dfrac{\sin Z}{Z}\right] = 1-1 = 0$

(29) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{x^2+y^2}$ Put $x^2+y^2=Z$

$= \lim\limits_{Z\to 0} \dfrac{1-\cos Z}{Z}$ $\left(\frac{0}{0}\right)$ apply L'hopitals rule

$= \lim\limits_{Z\to 0} \dfrac{\sin Z}{1} = 0.$

Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(30) $f(x,y) = \dfrac{x^4y^4}{(x^2+y^4)^3}$

along x axis, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \dfrac{x^4 \times 0}{(x^2+0)^3} = 0$

along $y=x$, $\lim\limits_{x\to 0} \dfrac{x^8}{(x^2+x^4)^3} = \lim\limits_{x\to 0} \dfrac{x^2}{(1+x^2)^3} = 0$

along $x = y^2$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{y\to 0} \dfrac{y^8 . y^4}{(2y^4)^3} = \dfrac{1}{8}$

hence limit doesn't exist.

(31) $f(x,y) = \dfrac{x-y^2}{x^2+y^2}$

along x axis, $\lim\limits_{x\to 0} \dfrac{x}{x^2} = \infty$.

along y axis, $\lim\limits_{y\to 0} \dfrac{-y^2}{y^2} = -1$

does not exist.

(33) $f(x,y) = \dfrac{x^2y^2}{x^4+y^4}$

along x axis, $\lim\limits_{x\to 0} \dfrac{x^2 \times 0}{x^4} = 0$

along $x=y$, $\lim\limits_{x\to 0} \dfrac{x^4}{x^4+x^4} = \dfrac{1}{2}$

doesn't exist.

(35) $f(x,y) = \begin{cases} \dfrac{x\,y^3}{x^2+y^6} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

Is f cts at $(0,0)$.

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$

along x axis, $\lim\limits_{x\to 0} \dfrac{x \times 0}{x^2} = 0$

along $x = y^3$, $\lim\limits_{y\to 0} \dfrac{y^6}{2y^6} = \dfrac{1}{2}$

$\therefore$ f is not cts ($\lim\limits_{(x,y)\to(0,0)} f$ does not exist).

(39) Given that

$f(x,y) = \begin{cases} \dfrac{3x^3 - 3y^3}{x^2-y^2} & , x^2 \neq y^2 \\ B & \text{other wise} \end{cases}$

is cts at $(0,0)$ find B.

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{\text{along } x \text{ axis}} f(x,y) = \lim\limits_{x\to 0} \dfrac{3x^3}{x^2} = 0$

$\therefore B = 0$. ($\because$ for f is cts at $(0,0)$ $\lim\limits_{(x,y)\to(0,0)} f(x,y) = B$).