Swaps

A swap is an agreement between two companies to exchange cash flows in future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated.

Example:- Suppose a company enters into a forward contract to buy 100 ounce of gold for \$ 1,200 per ounce in 1 year. The company can sell the gold as soon as it is received. The forward contract is therefore equivalent to a swap where the company agrees today, it will pay \$ 1,20,000 and receive 100 S, where S - price of Gold after 1 year.

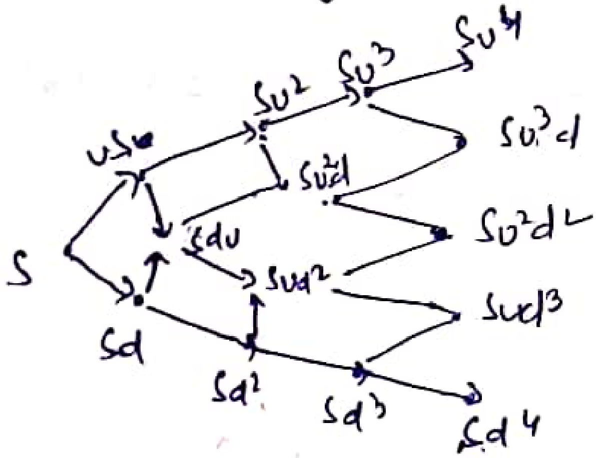
Mechanics of Interest rate swaps

The most common type of swap is a "plain vanilla" interest rate swap. In this swap a company agrees to pay cash flows equal to interest at a predetermined fixed rate on a notional principal for a predetermined number of years. In return, it receives interest at a floating rate (LIBOR) on the same notional principal for same period of time.

LIBOR → London Interbank Offered Rate.

Models of Asset Dynamics :-

1.) Binomial Lattice Model :- According to this model, if the price is known at the beginning of a period, the price at the beginning of the next period is one of the only two possible values (multiples of price at previous period) - a multiple u (for up) and a multiple to d (for down). Thus, if the price at the beginning of a period is S, it will be either uS or dS at the next period. The probabilities of these are p and $1-p$ respectively.



Accordingly, we define ν as the expected yearly growth rate

$$\nu = E[\ln(S_T/S_0)]$$

where S_0 is the initial stock price

and S_T is the price at the end of 1 year

define, σ as the yearly standard deviation.

$$\sigma^2 = Var[\ln(S_T/S_0)]$$

If a period length of Δt is chosen, then the parameters of the binomial lattice can be selected as

$$p = \frac{1}{2} + \frac{1}{2}\left(\frac{v}{\sigma}\right)\sqrt{\Delta t}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 1/u$$

Ex: Consider a stock with parameters $v = 15\%$ and $\sigma = 30\%$. We wish to make a binomial model based on weekly periods. then as $\Delta t = \frac{1}{52}$

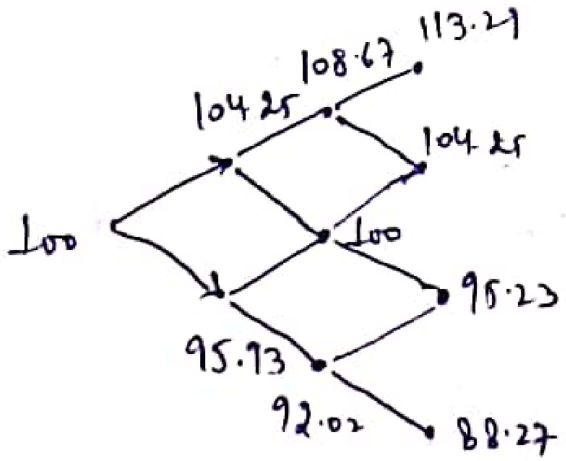
$$u = e^{0.30/\sqrt{52}} = 1.04248$$

$$v = 1/u = 0.95925$$

and

$$p = \frac{1}{2}\left(1 + \frac{0.15}{0.30} \times \sqrt{\frac{1}{52}}\right) = 0.534669.$$

Assuming $S(0) = 100$

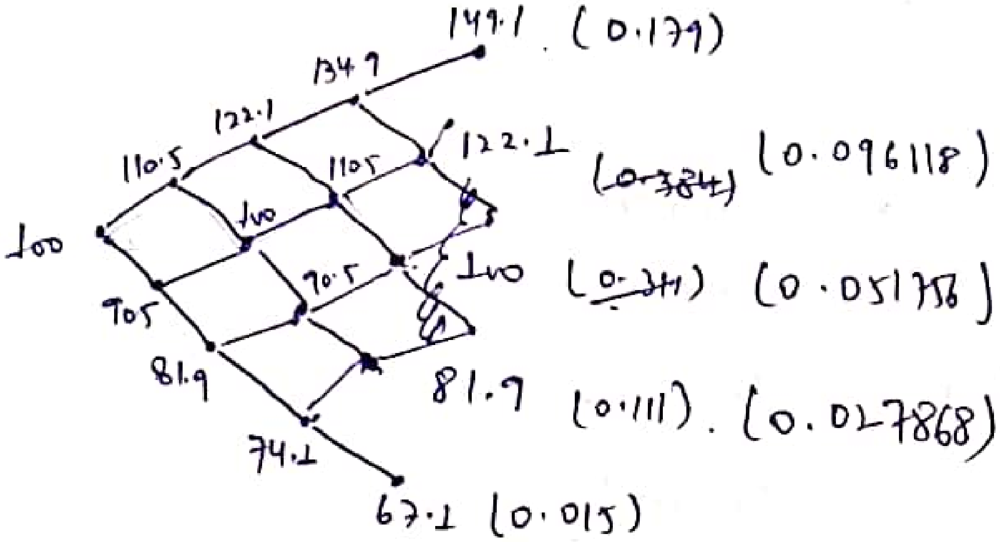


Q A stock with current value $S(0) = 100$ has an expected growth rate of its logarithm of $v = 12\%$ and volatility of that growth rate of $\sigma = 20\%$. find suitable parameters of a binomial lattice representing this stock with a basic elementary period of 3 months. Draw lattice & enter node values for 1 year.

$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\times\sqrt{1/4}} = 1.105$

$d = 1/u = 0.905$

$p = \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} = 0.65$



Multiplicative Model

The multiplicative model has form

$$S(k+1) = u(k)\,S(k) \qquad \text{—(1)}$$

for $k = 0, 1, \ldots, N-1$. Here $u(k)$ are mutually independent random variables.

The variable $u(k)$ defines the relative change in price between time k and $k+1$.

taking natural log in (1) we get

$$\ln S(k+1) = \ln S(k) + \ln u(k)$$

for $k = 0, 1, 2, \ldots, N-1$.

Now, let $w(k) = \ln u(k)$, for $k = 0, 1, \ldots, N-1$.

$w(k)$'s are normal random variables.

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We assume that they are mutually independent and that each has expected value $\bar{w}(k) = v$ and variance σ^2

$$u(k) = e^{w(k)} \quad \text{for } k = 0, 1, \ldots N-1$$

$u(k) \rightarrow$ is called lognormal random variable

Lognormal Prices

$$S(k) = u(k-1)\, S(k-1)$$
$$= u(k-1)\, u(k-2) \ldots u(0)\, S(0)$$

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} \ln u(i)$$
$$= \ln S(0) + \sum_{i=0}^{k-1} w(i)$$

If $\bar{w}(i) = v$ and variance σ^2 and are mutually independent

then $E[\ln S(k)] = \ln S(0) + vk$

$\text{var}[\ln S(k)] = k\sigma^2.$

- Suppose that w is normal and has expected value $\bar{w}$ and variance σ^2.

then $\bar{u} = e^{\bar{w} + \frac{1}{2}\sigma^2}$

Random walks and Wiener Processes

Suppose that we have N periods of length Δt, we define z by

$$z(t_{k+1}) = z(t_k) + \epsilon(t_k)\sqrt{\Delta t}$$

$$t_{k+1} = t_k + \Delta t$$

for $k = 0, 1, 2, \ldots, N$. This process is termed a random walk.

In these equations $\epsilon(t_k)$ is a normal random variables with mean 0 and variance 1. These random variables are mutually uncorrelated.

Now,

$$z(t_k) - z(t_j) = \sum_{i=j}^{k-1} \epsilon(t_i)\sqrt{\Delta t} \quad \text{for } j < k.$$

and $$E[z(t_k) - z(t_j)] = 0$$

and $$\operatorname{Var}[z(t_k) - z(t_j)] = E\left[\sum_{i=j}^{k-1} \epsilon(t_i)\sqrt{\Delta t}\right]^2$$

$$= E\left[\sum_{i=j}^{k-1} \epsilon(t_i)^2 \Delta t\right]$$

$$= (k-j)\Delta t = t_k - t_j$$

A Wiener process is obtained by taking the limit of the random walk process as $\Delta t \to 0$.

$$dz = \epsilon(t)\sqrt{dt}$$

where each $\epsilon(t)$ is a standardized normal random variable. The random variables $\epsilon(i)$ and $\epsilon(j)$ are uncorrelated whenever $i \neq j$.

We say process $z(t)$ is a Wiener process or Brownian motion if it satisfies the following:

(i) for any $s < t$, the quantity $z(t) - z(s)$ is a normal random variable with mean zero and variance $t-s$.

(ii) for any $0 \le t_1 \le t_2 \le t_3 \le t_4$, the random variables $z(t_2) - z(t_1)$ and $z(t_4) - z(t_3)$ are uncorrelated.

3.) $z(t_0) = 0$ with probability 1.

Generalized Wiener Processes and Ito Processes :-

$$dx(t) = a\,dt + b\,dz$$

where $x(t)$ is a random variable for each t, z is a Wiener process and a & b are constants.

$$x(t) = x(0) + at + b\,z(t)$$

Ito processes :-

$$dx(t) = a(x,t)\,dt + b(x,t)\,dz.$$

Stock Price Process :-

By multiplicative model.

$$\ln S(k+1) - \ln S(k) = w(k)$$

where $w(k)$'s are uncorrelated normal random variables.

Then the continuous-time expression is

$$d\ln S(t) = \nu\,dt + \sigma\,dz \quad — (*)$$

where ν and σ are constants and z is a Wiener process.

$$\ln S(t) = \ln S(0) + \nu t + \sigma z(t)$$

then $$E[\ln S(t)] = E[\ln S(0)] + \nu t$$

Hence, $E(\ln S(t))$ grows linearly with t.

This ~~must~~ process is termed geometric Brownian motion.

As, $d \ln[S(t)] = \frac{dS(t)}{S(t)}$

then * becomes

$$\frac{dS(t)}{S(t)} = \left(v + \frac{1}{2}\sigma^2\right) dt + \sigma dz$$

or $$dS(t) = \mu S(t) dt + \sigma S(t) dz$$

where $\mu = v + \frac{1}{2}\sigma^2$.

~~and~~ <u>Relations for geometric Brownian motion:-</u>

Suppose that geometric Brownian motion process $S(t)$ is governed by

$$dS(t) = \mu S(t) dt + \sigma S(t) dz.$$

Where z = standard Wiener process, $v = \mu - \frac{1}{2}\sigma^2$.

$$E[\ln[S(t)/S(0)]] = vt$$

$$\text{Stndev}[\ln[S(t)/S(0)]] = \sigma\sqrt{t}$$

$$E[S(t)/S(0)] = e^{\mu t}$$

$$\text{Stndev}[S(t)/S(0)] = e^{\mu t}(e^{\sigma^2 t} - 1)^{1/2}$$

Q1 A stock price is governed by geometric Brownian motion with $\mu = 0.20$ and $\sigma = 0.40$.

The initial price is $S(0) = 1$. $v = \mu - \frac{1}{2}\sigma^2$
$= 0.12$

Then $E[\ln S(1)] = E[\ln[S(1)/S(0)]] = vt = v \times 1$

$= 0.12$

$$Stdev[\ln S(1)] = Stdev[\ln[S(1)/S(0)]] =$$

$$= \sigma\sqrt{t} = \sigma\sqrt{1} = 0.40$$

$$E[S(1)] = e^{\mu t} = e^{\mu \times 1}$$

$$= e^{0.20} = 1.2214$$

$$Stdev[S(1)] = e^{\mu \times 1}[e^{\sigma^2 \times 1} - 1]^{1/2}$$

$$= 1.2214[$$

$$= 0.50877$$