# Chapter 3. Double and triple integrals

This material is covered in Thomas (chapter 15 in the 11th edition, or chapter 12 in the 10th edition).

*3.1 Remark.* What we will do is in some ways similar to integrals in one variable, definite integrals (which evaluate to a number) rather than indefinite integrals (which are essentially antiderivatives, and are functions).

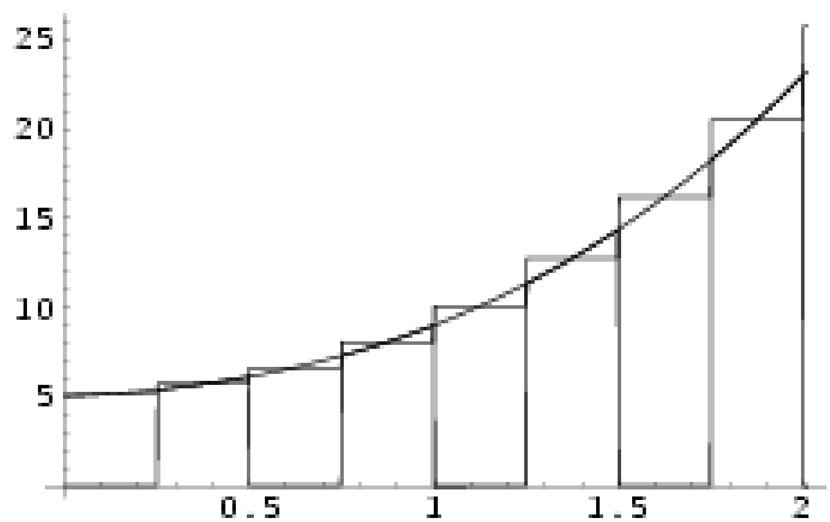
There are 3 ways to approach definite integrals $\int_a^b f(x)\,dx$ in one variable and we recall them briefly as we will develop something similar for functions $f(x,y)$ of two variables and $f(x,y,z)$ of three variables.

(i) $\int_a^b f(x)\,dx =$ a limit of certain Riemann sums

$$\lim \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where $[a,b]$ is divided into $n$ segments $[a,x_1] = [x_0,x_1]$, $[x_1,x_2], \ldots, [x_{n-1},x_n] = [x_{n-1},b]$ by choosing division points $a < x_1 < x_2 < \cdots < x_{n-1} < b$. By $\Delta x_i$ we mean the width $x_i - x_{i-1}$ of the $i$th interval. We also have to choose $c_1, c_2, \ldots, c_n$ with $x_{i-1} \le c_i \le x_i$ $(1 \le i \le n)$ before we can calculate the Riemann sum. We take the limit as the number $n$ of intervals tends to $\infty$ and the widths $\Delta x_i$ all tend to 0.

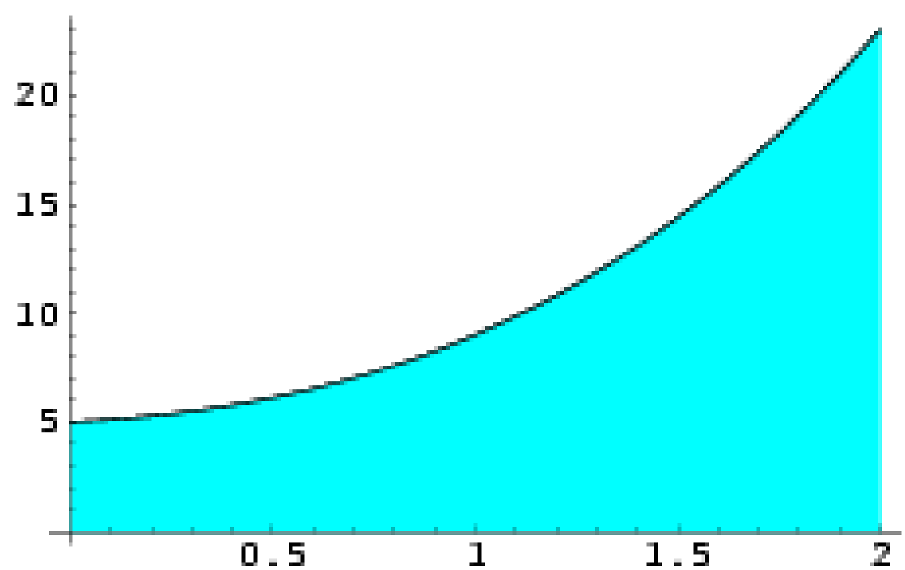
The idea is summarised by the following picture



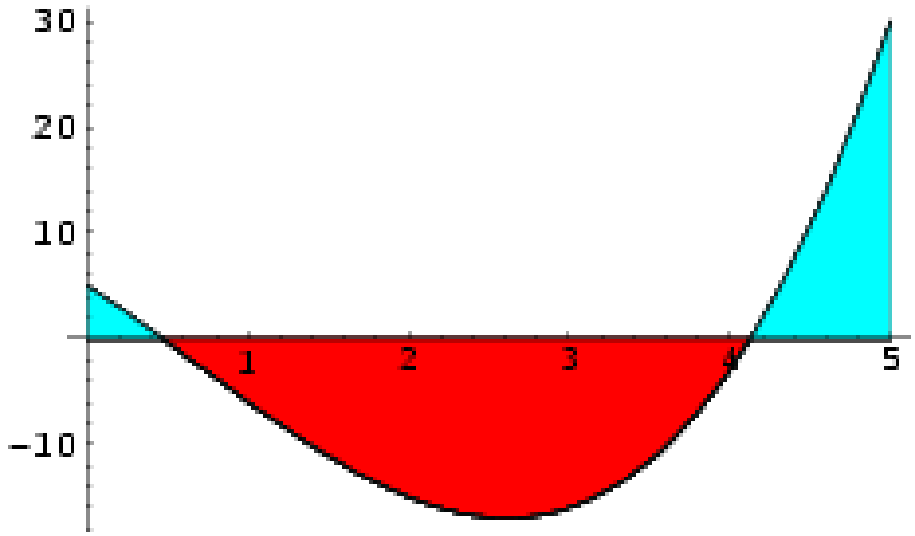

The Riemann sum corresponds to the sum of the widths times heights of the rectangles (and approximates the 'area' of the region under the curve, which represents the graph $y = f(x)$ for $a \le x \le b$). In this picture $n = 8$, all the widths $\Delta x_i$ are the same and the points $c_i$ are the middle points of the allowed interval for them.

The idea of the limit can be explained like this. We are likely to get a better approximation if we use more and narrower rectangles. It is a bit hard to really justify it properly, but the integral notation we use $\int_a^b f(x)\,dx$ is supposed to represent the limiting case where we have unbelievably narrow rectangles of width $dx$ (and then we need an unbelievably large number of them). The problem is we don't want to take $dx$ to be quite 0 as then the products $f(x)\,dx$ would be all 0. There is an explanation where we take the $dx$ to be so-called 'infinitesimal' numbers — that is smaller than any positive number you can imagine but somehow still not 0.

(ii) We can say graphically that $\int_a^b f(x)\,dx$ means the 'area' of the part of the plane between the graph $y = f(x)$ ($a \leq x \leq b$) and the $x$-axis.




This picture is good for case $f(x) \geq 0$. When $f(x) < 0$ some or all of the time, we count those parts of the region trapped between the graph and the $x$-axis where $f(x) < 0$ as negative (that is we subtract the total area where $f(x) < 0$ from the total area where $f(x) > 0$). If you look at the Riemann sums, you see $f(c_i)\Delta x_i$ and you'll see that comes out negative if $f(c_i) < 0$



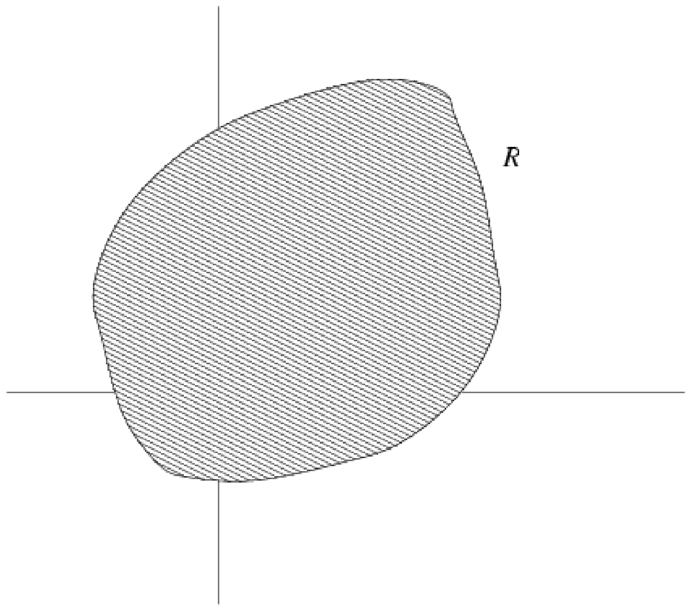

(iii) $\int_a^b f(x)\,dx = [F(x)]_{x=a}^b = F(b) - F(a)$ where $F(x)$ is an antiderivative of $f(x)$, that is a function where $F'(x) = f(x)$ $(a \leq x \leq b)$.

This connection of integrals with derivatives is so familiar that we are inclined to take it for granted. In fact it is an important result discovered early on in the subject, called the 'Fundamental Theorem of Integral Calculus' that makes the connection between limits of Riemann sums and antiderivatives.

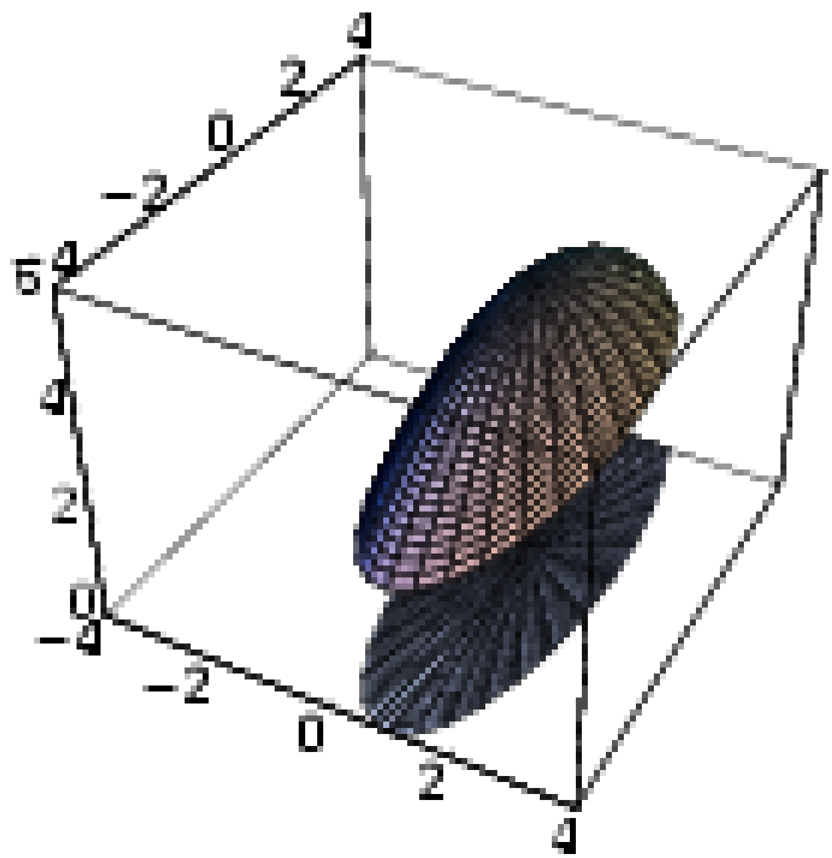
**3.2 Double integrals.** We are now going to give a brief definition of what a double integral is. We write double integrals as

$$\iint_R f(x,y)\,dx\,dy$$

where $f(x,y)$ is a function of tow variables that makes sense for $(x,y) \in R$, and $R$ is a part of the $(x,y)$-plane. We should not allow $R$ to be too complicated and we might picture $R$ as something like this:




We could visualise the situation via the graph $z = f(x, y)$ over the part of the horizontal plane indicated by $R$.



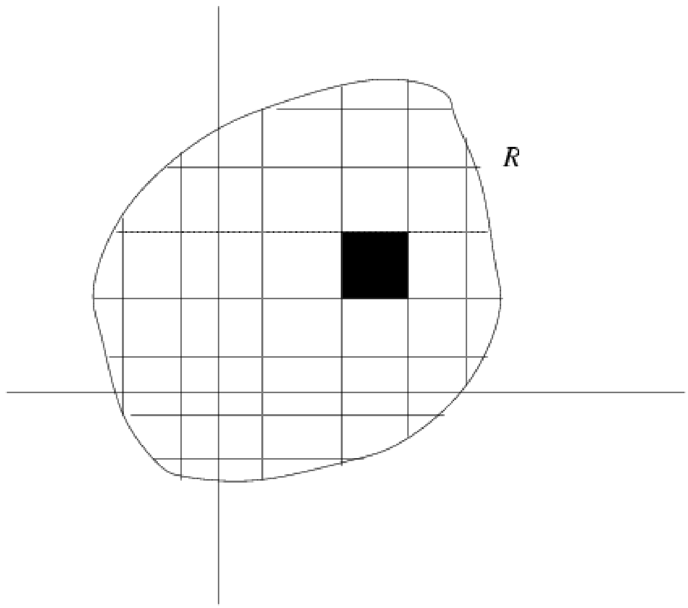
At least in the case where $f(x, y) \geq 0$ always, we can imagine the graph as a roof over a floor area $R$. The graphical interpretation of the double integral will be that it is the volume of the part of space under the roof. (So think of a wall around the perimeter of the floor area $R$, reaching up the the edge of the 'roof' or graph. At least in this picture the solid part of space enclosed by the floor, the walls and the roof would look like some sort of irregular cake.)

So this is the version for double integrals of the explanation in Remark 3.1 (ii) of ordinary single integrals. To make it more accurate, we have to cater for functions $f(x, y)$ that are sometimes negative, so that the 'roof' or graph is below the floor at times! In this case we have to subtract the volumes where the roof is below the floor, add the parts where the roof is above (that is the parts where $f(x, y) > 0$ are counted positive, and the parts where $f(x, y) < 0$ are counted negative).

So far what we have is a graphical or intuitive explanation, but its is not very useful if we want to compute the double integral. It is also unsatisfactory because it assumes we know what we mean by the volume and part of the reason for being at a loss for a way to compute the volume of such a strange 'cake' is that we don't have a very precise definition of what the volume is.

For single integrals this 'precise' explanation proceeds by the notion of a Riemann sum. Even if you did not think much of it, and indeed I'm sure there was not much attempt to cross the i's and dot the t's in the explanation of what Riemann sums are, and the explanation of how to deal with them, they are actually the basis for practical methods of calculating integrals numerically. (You've hear of the trapezoidal rule and Simpsons rule, which are based on rather similar ideas to Riemann sums. As engineers, you may well end up doing numerical simulations of designs and techniques like these are bread and butter of such simulations.)

Anyhow, to sketch how double integrals are really defined we need a two dimensional version of a Riemann sum. What we do is think in terms of dividing up the region $R$ (the floor in our picture) into a grid of rectangles (or squares). You could think of tiles on the floor, maybe better to think of very small tiles like mosaic tiles. We take the grid lines to be parallel to the $x$ and $y$ axes. Here is a picture of a grid, though this one has rather large rectangles.




One drawback we will see no matter how small we make the grid is that there will be irregular shapes at the edge. These do cause a bit of bother, but the idea is that if the grid is fine enough, the total effect of these incomplete rectangles around the edge should not matter much. Let's say we agree to omit any incomplete rectangles altogether.

No concentrate on just one of the grid rectangles (or tiles on our floor), like the one marked in black in the picture. The part of the total volume under the roof that is above this one rectangle will make a sort of pillar, and it will be the analogue for the three dimensional picture of the graph $z = f(x, y)$ of the rectangles we had in the Remark 3.1 (i) for functions of one variable.

You might look in the book to see nicely drawn artistic picture of what is going on here. We estimate the total volume under the 'roof' $z = f(x, y)$ by adding up the volumes of the pillars we have just described. We square off the top of each pillar to make it a flat topped pillar of a height $f(x, y)$ the same as the height of the graph at some point $(x, y)$ inside our chosen grid rectangle (or floor tile). We get the volume of the pillar as the height $f(x, y)$ times the area of the grid rectangle. If the side of the rectangle are $\Delta x$ along the $x$-direction and $\Delta y$ along the $y$-direction, that gives us a volume

$$f(x, y)\, \Delta x\, \Delta y$$

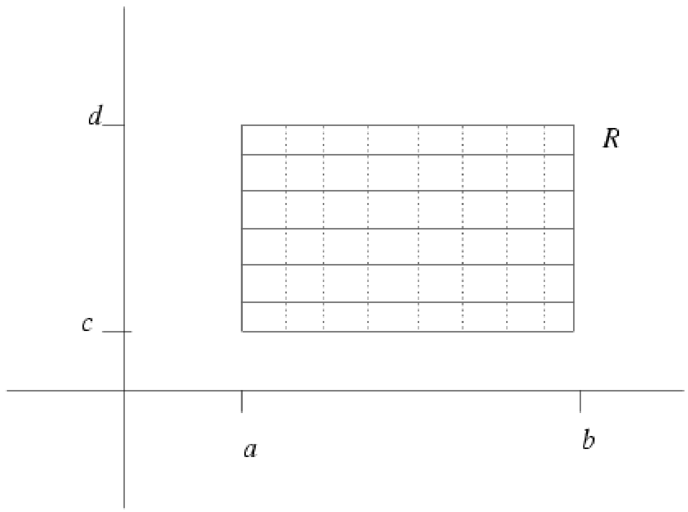
for that one pillar. We add all these up top get an estimate of the total volume 'under the graph'.

To get a better estimate, we should use more smaller grid rectangles. The integral is defined as the limit of the values we get by taking grids with smaller and smaller spacing between the grid lines.

**3.3 Fubini's theorem.** What we have so far is fairly theoretical or conceptual. It is in fact possible to make it practical with a computer program to calculate the sums we've been talking about, but for pencil and paper calculations we need something like Remark 3.1 (ii). The fact that works like that is called Fubini's theorem and we try to explain it first taking $R$ to be a rectangle with sides parallel to the axes: say

$$R = \{(x, y) \in \mathbb{R}^2 : a \le x \le b, c \le x \le d\}$$

(where $a < b$ and $c < d$ are numbers).




If we go back to thinking of the picture we had above, we now have a nice regular (rectangular) floor, though the roof above (the graph $z = f(x, y)$ could be very irregular. Or maybe we should think of a cake (or loaf of bread) baked in a rectangular cake tin. We might need to be a bit imaginative about how the cake rose so as to allow for very strange shapes for the top of the cake according to a graph $z = f(x, y)$.

Now, you could explain our earlier strategy to estimate the volume of the solid object like this. First slice the cake into thin slices one way (say along the $x$-direction first) and then slice it in the perpendicular direction. We end of with rectangular 'fingers' of cake and we add up the volumes of each one. But we estimate the volume of the finger by treating it as a box with a fixed height, even though it will in fact have a slightly irregular top. (Not too irregular though if we make many cuts.)

To get some sort of explanation of why Fubini's theorem works, we think of only slicing in one direction. Say we slice along the $x$-direction, perpendicular to the $y$-axis. That means $y$ is constant along each slice. Think of cutting up the bread into slices, preferably very thing slices. (In our picture, if we cut in the $x$-direction we will be cutting down the length of the bread, rather than across. We would be allowed to cut the other way, keeping $x$-fixed, if we wanted, but let's stick to the choice of cutting along the $x$-direction keeping $y$-fixed). We can thing then of a thin slice and we could imagine finding its volume by area and multiplying by its thickness.

Say we call the thickness $dy$ (for a very thin slice). How about the area of the face of the slice? Well we are looking at a profile of the slice as given by a graph $z = f(x, y)$ with $y$ fixed and $x$ varying between $a$ and $b$. So it has an area given by a single integral

$$\int_{x=a}^{x=b} f(x, y)\, dx$$

(where $y$ is kept fixed, or treated as a constant, while we do this integral). We had some similar ideas when dealing with $\partial f/\partial x$. We kept $y$ fixed and differentiated with respect to $x$. Or we took the graph of the function of one variable ($x$ variable in fact) that arises by chopping the graph

$z = f(x, y)$ in the $x$-direction, and looked at the slope of that. Anyhow, what we are doing now is integrating that function of $x$ rather than differentiating it.

The integral above is the area of one slice. To get its volume we should multiply by the thickness $dy$

$$\left( \int_{x=a}^{x=b} f(x, y)\, dx \right) dy$$

And then to get the total volume we should 'add' the volumes of the thin slices. Well, in the end we get an integral with respect to $y$ and what Fubini's theorem says is that

$$\boxed{\iint_R f(x, y)\, dx\, dy = \int_{y=c}^{y=d} \left( \int_{x=a}^{x=b} f(x, y)\, dx \right) dy}$$

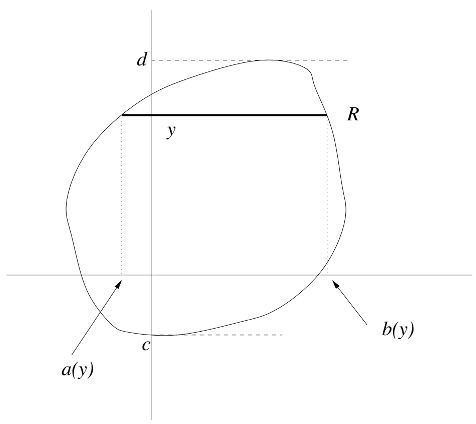
(when $R$ is the rectangle $R = [a, b] \times [c, d]$ we chose earlier).

Fubini's theorem also allows us to do the integral with respect to $y$ first (keeping $x$ fixed) and then the integral with respect to $x$.

$$\boxed{\iint_R f(x, y)\, dx\, dy = \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x, y)\, dy \right) dx}$$

In our explanation of why the theorem is true, this is what would happen if we sliced the other way. It may seem overkill to have two versions of the theorem, but there are examples where the calculations are much nicer if you do the $dx$ integral first that if you do the $dy$ integral first (and vice versa).

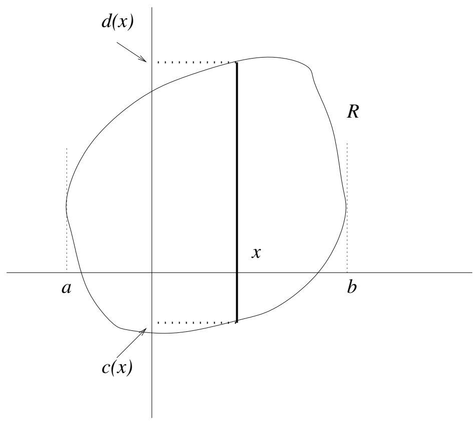
That was all for the case when $R$ is a rectangle, but a similar idea will work even when $R$ is more complicated. We can either integrate $dy$ first and then $dx$ or vice versa. But we need to take care that we cover all the points $(x, y)$ in $R$. If we keep $y$ constant at some value and then we have to figure out which $x$-values give $(x, y)$ inside $R$. That range of $x$-values will now depend on which $y$ we have fixed. In the picture that range of $x$ values is $a(y) \leq x \leq b(y)$.




$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \left( \int_{x=a(y)}^{x=b(y)} f(x,y)\,dx \right) dy$$

The range $c \leq y \leq d$ has to be so as to include all of $R$ (and no more).

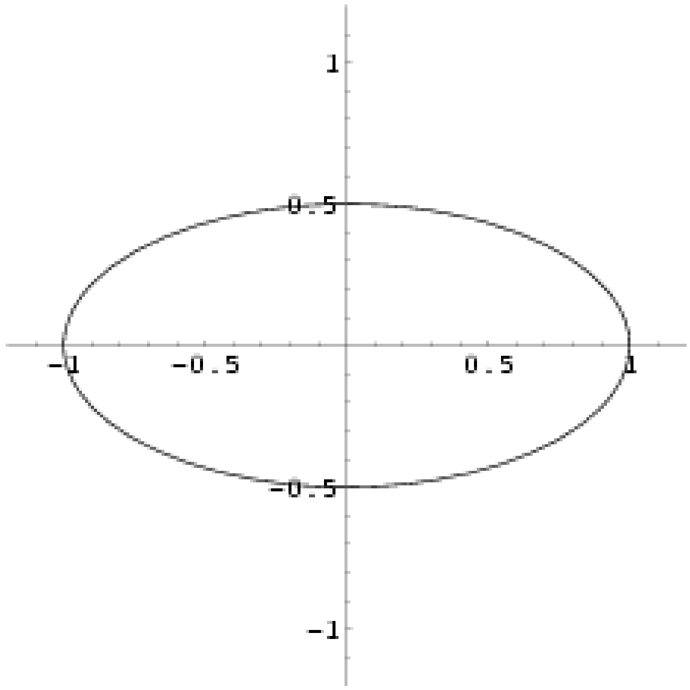
If we do the $dy$ integral first, keeping $x$ fixed, we need to figure out limits $c(x) \leq y \leq d(x)$ for $y$. The idea should be clear from the following picture and the formula below.




$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \left( \int_{y=c(x)}^{y=d(x)} f(x,y)\,dy \right) dx$$

*3.4 Example.* Find $\iint_R x^2\,dx\,dy$ when $R = \{(x,y) : x^2 + 4y^2 \leq 1\}$.

It could help to draw $R$ (it is the interior of an ellipse meeting the $x$-axis at $(\pm 1, 0)$ and the $y$-axis at $(0, \pm 1/2)$) but perhaps it is not really necessary to rely on the picture.




If we fix $y$ and we want to know which $x$-values to integrate over (for that $y$) it is is not hard to figure it out:

$$\begin{aligned} x^2 + 4y^2 &\le 1 \\ x^2 &\le 1 - 4y^2 \\ -\sqrt{1-4y^2} &\le \le \sqrt{1-4y^2} \end{aligned}$$

So the first integral to work out should be

$$\int_{x=-\sqrt{1-4y^2}}^{x=\sqrt{1-4y^2}} x^2\,dx.$$

Then we have to integrate that over all $y$ values for which there are any points. But it is fairly clear that there are points only when that square root is a square root of something that is positive. So we need

$$\begin{aligned} 1 - 4y^2 &\ge 0 \\ 1 &\ge 4y^2 \\ \frac{1}{4} &\ge y^2 \\ -\frac{1}{2} &\le y \le \frac{1}{2} \end{aligned}$$

We get

$$\begin{aligned}\iint_R x^2\,dx\,dy &= \int_{y=-1/2}^{y=1/2}\left(\int_{x=-\sqrt{1-4y^2}}^{x=\sqrt{1-4y^2}} x^2\,dx\right)dy\\ &= \int_{y=-1/2}^{y=1/2}\left[\frac{x^3}{3}\right]_{x=-\sqrt{1-4y^2}}^{x=\sqrt{1-4y^2}} dy\\ &= \int_{y=-1/2}^{y=1/2}\frac{2(1-4y^2)\sqrt{1-4y^2}}{3}\,dy\end{aligned}$$

Let $2y = \sin\theta$, $2\,dy = \cos\theta\,d\theta$. For $y = -1/2$, we get $-1 = \sin\theta$ and so $\theta = -\pi/2$. For $y = 1/2$, we get $1 = \sin\theta$ and so $\theta = \pi/2$.

So, after the substitution, the integral we want becomes

$$\begin{aligned}&\int_{\theta=-\pi/2}^{\theta=\pi/2}\frac{2}{3}(1-\sin^2\theta)\sqrt{1-\sin^2\theta}\,\frac{1}{2}\cos\theta\,d\theta\\ &= \int_{-\pi/2}^{\pi/2}\frac{1}{3}\cos^4\theta\,d\theta\\ &= \int_{-\pi/2}^{\pi/2}\frac{1}{3}\left(\frac{1}{2}(1+\cos 2\theta)\right)^2 d\theta\\ &= \frac{1}{12}\int_{-\pi/2}^{\pi/2}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta\\ &= \frac{1}{12}\int_{-\pi/2}^{\pi/2}\left((1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right)d\theta\\ &= \frac{1}{12}\int_{-\pi/2}^{\pi/2}\left(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right)d\theta\\ &= \frac{1}{12}\left[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_{-\pi/2}^{\pi/2} = \frac{\pi}{8}\end{aligned}$$

**3.5 Applications of Double integrals.**

(i) If we take the case of the constant function 1, $f(x, y) = 1$,

$$\begin{aligned}\iint_R f(x,y)\,dx\,dy &= \iint_R 1\,dx\,dy\\ &= \text{volume between the graph } z = 1 \text{ and}\\ &\quad\ \text{the region } R \text{ in the } x\text{-}y \text{ (horizontal) plane}\\ &= (\text{area of } R) \times (\text{height } = 1)\\ &= \text{area}(R)\end{aligned}$$

In short, we have

$$\boxed{\iint_R 1\,dx\,dy = \text{area}(R)}$$

(ii) If $R$ represents the shape of a thin (flat) plate, it will have a *surface density* $\sigma(x, y)$. By 'surface density' we mean mass per unit area and if the plate is the same thickness and the same material throughout that will be a constant

$$\sigma = \frac{\text{mass}}{\text{area}}.$$

But if it is not constant, we find the density at $(x, y)$ by taking a very tiny sample of the plate at $(x, y)$ and taking the mass per unit area of that. So if the sample is a rectangular piece of (infinitesimally short) sides $dx$ in the $x$-direction times $dy$ in the $y$-direction, then it has area $dx\,dy$. Say we call its mass $dm$ (for a tiny bit of mass), then the surface density is

$$\sigma(x, y) = \frac{dm}{dx\,dy}$$

To get the total mass we should add up the little small masses $dm$, or actually integrate them. We get

$$\boxed{\text{total Mass } = \iint_R dm = \iint_R \sigma(x, y)\,dx\,dy}$$

The centre of mass of such a plate is at a point $(\bar{x}, \bar{y})$ which is also computed with double integrals. The formulae are:

$$\boxed{\bar{x} = \frac{\iint_R x\,dm}{\text{mass}} = \frac{\iint_R x\sigma(x, y)\,dx\,dy}{\iint_R \sigma(x, y)\,dx\,dy}, \quad \bar{y} = \frac{\iint_R y\,dm}{\text{mass}} = \frac{\iint_R y\sigma(x, y)\,dx\,dy}{\iint_R \sigma(x, y)\,dx\,dy}}$$

where $\sigma(x, y)$ denotes the surface density as above. These formulae are arrived at by starting with the condition for the centre of mass to be along the $y$-axis (so $\bar{x} = 0$). The condition is a moment condition, that the total moment of the body around the axis should be 0. The total moment about the $y$-axis is $\iint_R x\,dm$. We will not go into detail on this derivation of the formula for $\bar{x}$.

**3.6 Triple integrals.** We now give a brief explanation of what a triple integral

$$\iiint_D f(x, y, z)\,dx\,dy\,dz$$

of a function $f(x, y, z)$ of 3 variables over a region $D$ in $\mathbb{R}^3$ (space) means.

As we already know we cannot visualise the graph of a function of 3 variables in any realistic way (as it would require looking at 4 dimensional space) we cannot be guided by the same sort of graphical pictures as we had in the case of single integrals and double integrals. (Remember Remark 3.1 (ii) and the 'volume under the graph' idea of 3.2.)

We can proceed to think of Riemann sums, without the graphical back up. We think of $D$ as a shape (solid shape) in space and we imagine chopping $D$ up into a lot of small boxes with sides parallel to the axes. Think of slicing $D$ first parallel to the $x$-$y$ plane, then parallel to the $y$-$z$ plane and finally parallel to the $x$-$z$ plane. Preferably we should use very thin separations between the places where we make the slices, but we have to balance that against the amount of work we have to do if we end up with many tiny boxes.

Look at one of the boxes we get from $D$. Say it has a corner at $(x, y, , z)$ and side lengths $\Delta x$, $\Delta y$ and $\Delta z$. We take the product

$$f(x, y, z)\, \Delta x\, \Delta y\, \Delta z = f(x, y, z) \times \text{ Volume.}$$

We do this for each little box and add up the results to get a Riemann sum

$$\sum f(x, y, z)\, \Delta x\, \Delta y\, \Delta z$$

(We should agree some policy about what to do with the edges. If there are boxes that are not fully inside $D$ we could agree to exclude them from the sum, say.) We now define the integral

$$\iiint_D f(x, y, z)\, dx\, dy\, dz = \lim \text{(Riemann sums)}$$

where we take the limit as the Riemann sums get finer and finer (in the sense that the maximum side length of the little boxes is becoming closer and closer to 0). If we are lucky, this limit makes sense — and there is a theorem that says the limit will make sense if $D$ is bounded, not too badly behaved and $f$ is a continuous function. So we will take it that the limit makes sense.

As for a picture to think about, we can think maybe of the example we used of mass. If $f(x, y, z)$ is the *density* (= mass per unit volume) of a solid object occupying the region $D$ of space, then

$$f(x, y, z)\, dx\, dy\, dz = \text{density } \times \text{ volume } = \text{ mass } = dm$$

(mass of a little tiny piece of the solid) and we get the total mass by adding these up. So

$$\iiint_D \text{(density function)}\, dx\, dy\, dz = \text{ total mass.}$$

At least in the case where $f(x, y, z) \geq 0$ always, we can interpret triple integrals this way.

An even simpler situation is where we take $f(x, y, z) \equiv 1$ to be the constant function 1. Then

$$f(x, y, z)\, dx\, dy\, dz = dx\, dy\, dz = \text{ volume } = dV$$

is just the volume of the little tiny piece. When we add these up (or integrate the constant function 1) we get

$$\iiint_D 1\,dx\,dy\,dz = \text{ total volume.}$$

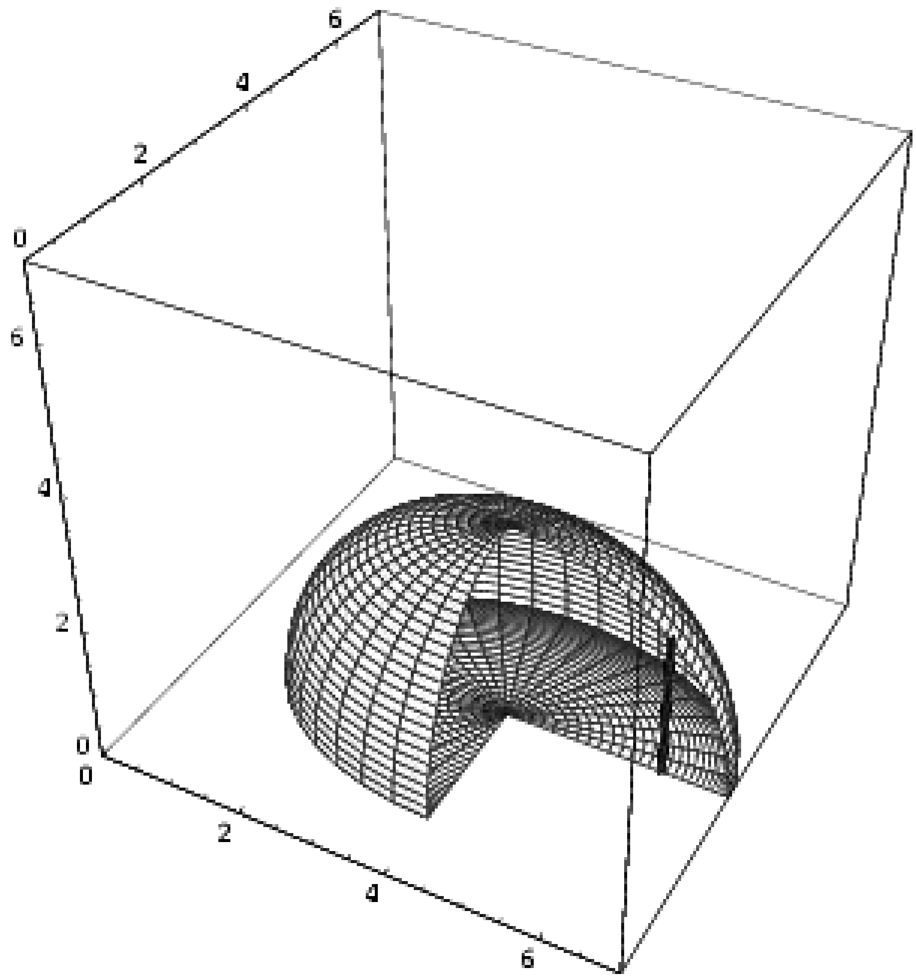
As for double integrals, there is a **Fubini Theorem** for triple integrals that allows us to work out $\iiint_D f(x,y,z)\,dx\,dy\,dz$ by working out three single integrals. The first integral (or inner integral) should be with respect to one of the variables, keeping the other two constant. Say we integrate $dz$ first, keeping $(x,y)$ fixed. We should integrate over all $z$ that give points $(x,y,z) \in D$. At least in simple cases, that will be a range of $z$ from a smallest we might call $z_0(x,y)$ to a largest $z_1(x,y)$.

In the following picture, there is an attempt to explain this. The curved surface is in fact the part of the ellipsoid

$$\frac{(x-4)^2}{3^2} + \frac{(y-2)^2}{2^2} + \frac{z^2}{3^2} = 1$$

where $z \geq 0$ and the flat part at the base is where $z = 0$. A quarter is cut away to allow us to see inside the object

$$D = \left\{(x,y,z) : \frac{(x-4)^2}{3^2} + \frac{(y-2)^2}{2^2} + \frac{z^2}{3^2} \leq 1, z \geq 0\right\}$$




For each fixed $(x, y)$ (like the one shown) we have to find the limits for $z$ so that $(x, y, z) \in D$ and it is easy enough to see that the smallest $z$ is $z_0(x, y) = 0$, while the largest is

$$z_1(x, y) = 3\sqrt{1 - \frac{(x-4)^2}{3^2} - \frac{(y-2)^2}{2^2}}$$

obtained by solving for $z$ in terms of $(x, y)$ when $(x, y, z)$ is on the upper (curved) surface.

We can see that the values of $(x, y)$ for which there are any possible $z$ to worry about are those $(x, y)$ where the square root is a square root of something positive. So, those $(x, y)$ where

$$\frac{(x-4)^2}{3^2} + \frac{(y-2)^2}{2^2} \leq 1.$$

Another way to think of it is that this is the outline of the object when viewed along the $z$-direction (from far away).

Anyhow our first integral (if we integrate $dz$ first) is

$$\int_{z=0}^{z=3\sqrt{1-(x-4)^2/9-(y-2)^2/4}} f(x, y, z)\, dz$$

Say we next integrate $dx$, keeping $y$ fixed. The inequality to be satisfied by all the $(x, y)$ we need to worry about is above and it can be expressed as

$$-3\sqrt{1 - \frac{(y-2)^2}{2^2}} \leq x - 4 \leq 3\sqrt{1 - \frac{(y-2)^2}{2^2}}$$

so that our next integral should be

$$\int_{x=4-3\sqrt{1-(y-2)^2/4}}^{x=4+3\sqrt{1-(y-2)^2/4}} \left( \int_{z=0}^{z=3\sqrt{1-(x-4)^2/9-(y-2)^2/4}} f(x,y,z)\,dz \right) dx$$

Finally we have to integrate this $dy$. The limits for $y$ are those corresponding to the extreme values of $y$ for points in $D$. In this case the restriction on $y$ is

$$1 - \frac{(y-2)^2}{2^2} \geq 0$$

and that turns out to be the same as

$$-2 \leq y - 2 \leq 2$$

or

$$0 \leq y \leq 4$$

So in this case $\iiint_D f(x,y,z)\,dx\,dy\,dz$ is

$$\int_{y=0}^{y=4} \left( \int_{x=4-3\sqrt{1-(y-2)^2/4}}^{x=4+3\sqrt{1-(y-2)^2/4}} \left( \int_{z=0}^{z=3\sqrt{1-(x-4)^2/9-(y-2)^2/4}} f(x,y,z)\,dz \right) dx \right) dy$$

It is possible to do the integrals in a different order. Say $dy$ first, then $dx$ and finally $dz$. All the limits will be changed if we do that and we would get

$$\int_{z=0}^{z=3} \left( \int_{x=4-3\sqrt{1-z^2/9}}^{x=4+3\sqrt{1-z^2/9}} \left( \int_{y=2-2\sqrt{1-(x-4)^2/9-z^2/9}}^{y=2+2\sqrt{1-(x-4)^2/9-z^2/9}} f(x,y,z)\,dy \right) dx \right) dz$$

An advantage of being able to choose the order is that sometimes the calculations are easier in one order than another. A trick that is sometimes useful for working out an iterated integral like the one we have just written is this:- figure out which $D \subset \mathbb{R}^3$ it corresponds to, write the integral as $\iiint_D f(x,y,z)\,dx\,dy\,dz$, that is use Fubini's theorem in reverse first, and then work out $\iiint_D f(x,y,z)\,dx\,dy\,dz$ with Fubini's theorem using a different order for the single integrals. Sometimes it turns a hard problem into one that is easier.

**3.7 Change of variables in multiple integrals.** We now come to a topic that goes by the name 'substitution' in the case of ordinary single integrals. You may recall that substitution for functions of one variable can be justified using the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \text{ when } y = y(u), u = u(x), y = y(u(x)).$$

For indefinite integrals it allows us to say that

$$\int g(u(x))\,dx = \int g(u)\frac{dx}{du}\,du$$

if we interpret the right hand side, a function of $u$, as a function of $x$ by $u = u(x)$. For definite integrals we can change limits as well as variables and get an equation that says two numbers are equal.

$$\int_{x=a}^{x=b} g(u(x))\,dx = \int_{u=u(a)}^{u=u(b)} g(u)\frac{dx}{du}\,du.$$

The point to remember from this is that when we change from integrating over the interval $[a, b]$ in $x$, we must not only change the range of integration to the corresponding range in the $u$ variable, we must also multiply the integrand by a factor $dx/du$.

For integrals in two variables (and similarly in three variables) we have to explain what that factor is that works in a similar way. It is the absolute value of a certain determinant of partial derivatives. Suppose we change from $(x, y)$ to $(u, v)$ say, we have to change $dx\,dy$ into a multiple of $du\,dv$ and the multiple is

$$\left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}\right|$$

The matrix inside the determinant is called a *Jacobian matrix*. Its rows are the gradient vectors of $x$ and $y$ with respect to the $u$ and $v$ variables. The determinant itself is called a *Jacobian determinant*.

The rule is then that we must change $dx\,dy$ to

$$dx\,dy = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right|\,du\,dv$$

To change an integral

$$\iint_R f(x, y)\,dx\,dy$$

to an integral in $(u, v) = (u(x, y), v(x, y))$, we have to change $R$ to the same set described in the $(u, v)$ variables and change $dx\,dy$ as above.

For the case of triple integrals, if we change from $(x, y, z)$ coordinates to

$$(u, v, w) = (u(x, y, z), v(x, y, z), w(x, y, z))$$

we have to make a similar change

$$dx\,dy\,dz = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}\right|\,du\,dv\,dw$$

While this theory can be applied to any change of coordinates, the ones that appear most often are polar coordinates in the plane, and cylindrical and spherical coordinates in space. So we work out what these Jacobian factors are in polar coordinates to start with. We can relate cartesian $(x, y)$ coordinates in $\mathbb{R}^2$ to polar coordinates $(r, \theta)$ via

$$x = r\cos\theta, \quad y = r\sin\theta$$

and so we can work out the 4 partial derivatives we need for the Jacobian.

$$\begin{aligned} \frac{\partial x}{\partial r} &= \cos\theta \\ \frac{\partial x}{\partial \theta} &= -r\sin\theta \\ \frac{\partial y}{\partial r} &= \sin\theta \\ \frac{\partial y}{\partial \theta} &= r\cos\theta \end{aligned}$$

Then the Jacobian matrix is

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

The determinant is

$$r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$$

The absolute value of the determinant is also $r$ (as long as we keep $r > 0$ as we usually do for polar coordinates). This gives us the relation

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

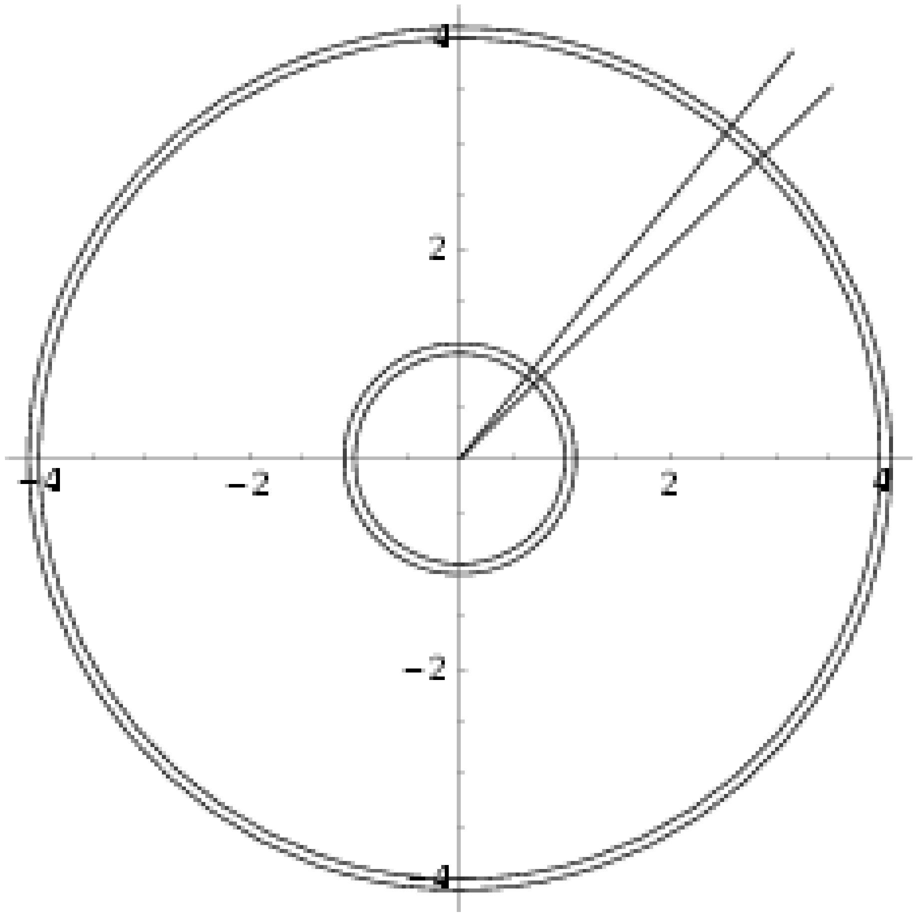
For the change from cartesian $(x, y, z)$ coordinates in space to cylindrical $(r, \theta, z)$ coordinates the calculation is not really much harder that what we have just done. We won't give the details but it comes to

$$\boxed{dx\,dy\,dz = r\,dr\,d\theta\,dz}$$

For spherical coordinates $(\rho, \theta, \phi)$ in space, the calculation is a little longer. Again we will not work it out but just give the result:

$$\boxed{dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

As we have not given any details on how these Jacobian factors are justified, we look at a picture relating to polar coordinates in the plane. The inner circle has radius 1, the next has radius 1.1, the next 4 and the outer one 4.1. The rays are at $\pi/4$ and $\pi/4 + 0.1$.




You can see that the area of the 'polar rectangle' at $(r, \theta) = (1, \pi/4)$ is smaller than the area of the polar rectangle at $(4, \pi/4)$. Both rectangles of squares of side 0.1 in polar coordinates, but it is reasonably clear that the outer one has 4 times the area of the inner one. In fact a polar rectangle with one corner at $(r, \theta)$ and opposite corner at $(r + dr, \theta + d\theta)$ is a (slightly bent) rectangle in the plane with side lengths $dr$ and $r\,d\theta$. This is a way to see that the $r\,dr\,d\theta$ formula is at least plausible.

*3.8 Example.* Find

$$\iint_R \left(8 - \frac{(x^2+y^2)^2}{2}\right)\,dx\,dy$$

where $R = \{(x, y) : x^2 + y^2 \leq 2\}$.

This is a problem that works out rather more easily in polar coordinates. We can describe $R$ in polar coordinates as the points $(r, \theta)$ with $0 \leq r \leq \sqrt{2}$ and $0 \leq \theta < 2\pi$. In this way we can express the integral as

$$\iint \left(8 - \frac{(r^2)^2}{2}\right) r\,dr\,d\theta$$

Putting in the limits

$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi}\left(\int_{r=0}^{r=\sqrt{2}}\left(8-\frac{(r^2)^2}{2}\right)r\,dr\right)d\theta &= \int_{\theta=0}^{\theta=2\pi}\left(\int_{r=0}^{r=\sqrt{2}} 8r-\frac{r^5}{2}\,dr\right)d\theta \\
&= \int_{\theta=0}^{\theta=2\pi}\left[4r^2-\frac{r^6}{12}\right]_{r=0}^{r=\sqrt{2}} d\theta \\
&= \int_{\theta=0}^{\theta=2\pi}\left(8-\frac{8}{12}\right)d\theta \\
&= \frac{23}{3}(2\pi)=\frac{46\pi}{3}
\end{aligned}$$

**3.9 Applications of triple integrals.** In fact we have already mentioned two applications (mass and volume) but we repeat them here so as to have them on one place. Then there is centre of mass (the formulae look similar to the ones we had for thin plates, but the 3 variable ones are more realistic since massive objects are usually 3 dimensional). Finally we mention moments of inertia.

(i) If $D \subset \mathbb{R}^3$, then

$$\iiint_D 1\,dx\,dy\,dz = \text{volume}(D)$$

(ii) If $D \subset \mathbb{R}^3$ and $f(x,y,z)$ makes sense for $(x,y,z) \in D$, then the *average value* of the function $f$ over $D$ is defined as follows:

$$\text{average}_D(f) = \frac{\iiint_D f(x,y,z)\,dx\,dy\,dz}{\text{volume}(D)} = \frac{\iiint_D f(x,y,z)\,dx\,dy\,dz}{\iiint_D 1\,dx\,dy\,dz}$$

(iii) If a solid object occupies a region $D \subset \mathbb{R}^3$ and has (possibly variable) density $\delta(x,y,z)$ at $(x,y,z) \in D$, then

$$\iiint_D \delta(x,y,z)\,dx\,dy\,dz = \iiint_D dm = \text{mass of the object}$$

The *centre of mass* of the solid is the point $(\bar{x},\bar{y},\bar{z})$ given by

$$\bar{x} = \frac{\iiint_D x\delta(x,y,z)\,dx\,dy\,dz}{\text{mass}}, \quad \bar{y} = \frac{\iiint_D y\delta(x,y,z)\,dx\,dy\,dz}{\text{mass}},$$
$$\bar{z} = \frac{\iiint_D z\delta(x,y,z)\,dx\,dy\,dz}{\text{mass}}.$$

The *moment of inertia* $I$ of a solid object about a fixed axis is a quantity that takes the place of mass when we deal with rotating the object about a fixed axis as opposed to dealing with unconstrained motion. It measures the resistance of the body to being set in rotational motion about the axis in the same way that mass measure the resistance of a body to being pushed along. One manifestation of that is in the formula for kinetic energy. For arbitrary motion of a mass $m$ with velocity vector $\mathbf{v}$ the kinetic energy is $\frac{1}{2}m\|\mathbf{v}\|^2$. For rotational motion of a body around an axis, the different particles making up the body are rotating in unison and all have a common angular velocity $\omega$ (= rate of change of angle around the axis, $\omega = \frac{d\theta}{dt}$). Those close to the axis have a smaller actual velocity than those farther away (since the farther away ones go around a big circle in the same time the closer in points go around a small circle). It comes down to a relation

$$\|\mathbf{v}\| = r_\perp(x, y, z)\omega$$

where $r_\perp(x, y, z)$ stands for the perpendicular distance from the point $(x, y, z)$ in the body to the axis. The formula for $I$ is then

$$\text{moment of inertia about axis} \\ = I = \iiint_D (r_\perp(x, y, z))^2 \, dm = \iiint_D (r_\perp(x, y, z))^2 \delta(x, y, z) \, dm$$

As a special case we can work out the moments of inertia about the $x$-, $y$- and $z$-axes. We get

$$\begin{aligned} I_x &= \iiint_D (y^2 + z^2) \, dm = \iiint_D (y^2 + z^2)\delta(x, y, z) \, dm \\ I_y &= \iiint_D (x^2 + z^2) \, dm = \iiint_D (y^2 + z^2)\delta(x, y, z) \, dm \\ I_z &= \iiint_D (x^2 + y^2) \, dm = \iiint_D (y^2 + z^2)\delta(x, y, z) \, dm \end{aligned}$$

There is a theorem that allows one to work out the moment of inertia about any axis though the centre of mass in terms of $I_x$, $I_y$, $I_z$ and 3 other numbers that are also given by integrals. These other numbers are denoted $I_{xy}$, $I_{xz}$ and $I_{xy}$ and are given by

$$\begin{aligned} I_{xy} &= -\iiint_D xy\,dm = -\iiint_D xy\delta(x,y,z)\,dm \\ I_{xz} &= -\iiint_D xz\,dm = -\iiint_D xz\delta(x,y,z)\,dm \\ I_{yz} &= -\iiint_D yz\,dm = -\iiint_D yz\delta(x,y,z)\,dm \end{aligned}$$

The 6 numbers are usually arranged as a $3 \times 3$ matrix

$$\begin{pmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{pmatrix}$$

which is called the inertia matrix or inertia tensor of the body.

We need to assume that the origin is the centre of mass. Then we can work out the moment of inertia of the body about any axis using the inertia matrix. The formula will not concern us, but here it is anyhow. If $\mathbf{u} = (u_1, u_2, u_3)$ is a unit vector parallel to the axis then the moment of inertia is

$$(u_1\,u_2\,u_3) \begin{pmatrix} I_x & I_{xy} & I_{xz} \\ I_{xy} & I_y & I_{yz} \\ I_{xz} & I_{yz} & I_z \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

This matrix product works out as a number (or $1 \times 1$ matrix)

$$I_x u_1^2 + I_y u_2^2 + I_z u_3^2 + 2I_{xy}u_1u_2 + 2I_{yz}u_2u_3 + 2I_{xz}u_1u_3$$

This seems to link in to linear algebra and it does. The eigenvectors of the inertia matrix are called 'principal axes'. Except in cases where the body is exceptionally symmetric, there will be just 3 such axes. They have the following physical significance: if the body is rotated about any axis other than a principal axis, then it will tend to vibrate at higher rates of rotation. This applies to wheels, shafts, rotors in electric motors, propellers, fan blades and anything that rotates. It is important that the wheel (or whatever it is) should be designed so that the axis of rotation is one of the principal axes. If you've been at a tyre shop when they change a tyre, you'll notice that they will clip on small lead weights around the rim to 'balance' the wheel. These weights are actually changing the inertia matrix slightly to compensate for irregularities in the wheel rim, or the tyre, or the fact that the valve stem has a small effect. A wheel rim (or a propeller) is not fully symmetric under rotation about its axle. There are usually air holes or other shaping to the design. These have to be designed so as to keep the axle as a principal axis.

So Mathematical things have their uses!

*3.10 Examples.* (i) Find the volume of a solid ball of radius $a$.

This is a problem that is well suited to an integral in spherical coordinates. We can take the ball to be centered at the origin, so that it is

$$D = \{(x, y, z) \in \mathbb{R}^3 : \sqrt{x^2 + y^2 + z^2} \leq a\}.$$

We know

$$\text{volume}(D) = \iiint_D 1 \, dx \, dy \, dz.$$

Although this integral is not quite impossible if we do it in $x$-$y$-$z$ coordinates via Fubini's theorem, the calculations are fairly long. Using spherical coordinates we can describe $D$ as

$$\{(\rho, \theta, \phi) : 0 \leq \rho \leq a, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi\}$$

and the integral for the volume becomes

$$\begin{aligned}
\text{volume} &= \int_{\phi=0}^{\phi=\pi} \left( \int_{\theta=0}^{\theta=2\pi} \left( \int_{\rho=0}^{\rho=a} \rho^2 \sin\phi \, d\rho \right) d\theta \right) d\phi \\
&= \int_{\phi=0}^{\phi=\pi} \left( \int_{\theta=0}^{\theta=2\pi} \left[ \frac{\rho^3}{3} \sin\phi \right]_{\rho=0}^{\rho=a} d\theta \right) d\phi \\
&= \int_{\phi=0}^{\phi=\pi} \left( \int_{\theta=0}^{\theta=2\pi} \frac{a^3}{3} \sin\phi \, d\theta \right) d\phi \\
&= \int_{\phi=0}^{\phi=\pi} \left( 2\pi \frac{a^3}{3} \sin\phi \right) d\phi \\
&= \left[ -2\pi \frac{a^3}{3} \cos\phi \right]_{\phi=0}^{\phi=\pi} \\
&= -2\pi \frac{a^3}{3} \cos\pi - (-2\pi \frac{a^3}{3} \cos 0) \\
&= -2\pi \frac{a^3}{3}(-1) + 2\pi \frac{a^3}{3} = \frac{4}{3}\pi a^3
\end{aligned}$$

Of course this is the answer we know all along.

(ii) Find the volume of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$$

We want the triple integral of the constant function 1 over the part of space defined above. This is a case where an unusual change of variables saves a lot of pain. Say we define

$$u = \frac{x}{a}, \quad v = \frac{y}{b}, \quad w = \frac{z}{c}.$$

In $(u, v, w)$ coordinates we describe the ellipse as

$$u^2 + v^2 + w^2 \leq 1$$

(that is a unit ball centre the origin).

To change the integral for the volume

$$\text{volume(ellipsoid)} = \iiint_{\text{ellipsoid}} 1 \, dx \, dy \, dz$$

into an integral in $du \, dv \, dw$ we need to work out the Jacobian determinant. Using $x = au$, $v = ay$, and $z = cw$ we get

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

The determinant of this matrix is $abc$ and the absolute value of the determinant is also $abc$. So

$$\begin{aligned} \iiint_{\text{ellipsoid}} 1 \, dx \, dy \, dz &= \iiint_{u^2+v^2+w^2 \leq 1} abc \, du \, dv \, dw \\ &= abc \iiint_{u^2+v^2+w^2 \leq 1} 1 \, du \, dv \, dw. \end{aligned}$$

We can bring the factor $abc$ outside the integral because it is a constant. This latter integral is the volume of the unit ball, and so it is $\frac{4}{3}\pi$. So the volume of the ellipse is

$$\frac{4}{3}\pi abc$$

(iii) Find the centre of mass of a uniform solid half ball.

We can take the half ball to be the top hemisphere of a ball of some radius $a$ centered at the origin. So we can take the region of space occupied by the solid to be

$$D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq a^2, z \geq 0\}$$

The radius $a$ will have to be part of the answer. Since we are told that the object is uniform, it means that the density $\delta(x, y, z)$ is a constant $\delta$ throughout the object. That tell us that the value of $\delta$ will not come into the answer. For example

$$\begin{aligned}
\bar{z} &= \frac{\iiint_D z\delta(x,y,z)\,dx\,dy\,dz}{\text{mass}}. \\
&= \frac{\delta \iiint_D z\,dx\,dy\,dz}{\iiint_D \delta\,dx\,dy\,dz} \\
&= \frac{\delta \iiint_D z\,dx\,dy\,dz}{\iiint_D \delta\,dx\,dy\,dz} \\
&= \frac{\iiint_D z\,dx\,dy\,dz}{\iiint_D 1\,dx\,dy\,dz} \\
&= \frac{\iiint_D z\,dx\,dy\,dz}{\text{volume}}
\end{aligned}$$

This fact (that the density does not come in to the position of the centre of mass in the case of constant density) is always true, but we don't really need it. We could just work out $\bar{z}$ and find that the value of $\delta$ cancels out in the end.

In the case of $\bar{x}$ and $\bar{y}$, we have a similar fact that $\delta$ does not come into their values. So the position of $(\bar{x}, \bar{y}, \bar{z})$ depends only on the shape of the object if it is made of a uniform material. But the symmetry of the object around the $z$-axis then makes it fairly obvious that the centre of mass is along the $z$-axis. In other words $\bar{x} = \bar{y} = 0$ in this case. We could go about working out the integrals for $\bar{x}$ and $\bar{y}$, but if we do it right we will get 0 both times. So really the question is to find $\bar{z} =$ the distance the centre of mass is away from the centre along the central axis of the half-ball. We'll do this in spherical coordinates $(\rho, \theta, \phi)$ and we need the ranges $0 \leq \rho \leq a$, $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi/2$ to cover the half-ball. Recall

$$z = \rho\cos\phi, \quad dx\,dy\,dz = \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$$

and so we get

$$\begin{aligned}
\iiint_D z\,dx\,dy\,dz &= \iint (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\
&= \int_{\rho=0}^{\rho=a}\left(\int_{\theta=0}^{\theta=2\pi}\left(\int_{\phi=0}^{\phi=\pi/2}\rho^3\cos\phi\sin\phi\,d\rho\right)\,d\theta\right)\,d\phi \\
&= \int_{\rho=0}^{\rho=a}\left(\int_{\theta=0}^{\theta=2\pi}\left(\int_{\phi=0}^{\phi=\pi/2}\frac{\rho^3}{2}\sin 2\phi\,d\rho\right)\,d\theta\right)\,d\phi \\
&= \int_{\rho=0}^{\rho=a}\left(\int_{\theta=0}^{\theta=2\pi}\left(\left[-\frac{\rho^3}{4}\cos 2\phi\right]_{\phi=0}^{\phi=\pi/2}\right)\,d\theta\right)\,d\phi \\
&= \int_{\rho=0}^{\rho=a}\left(\int_{\theta=0}^{\theta=2\pi}\left(\frac{\rho^3}{2}\right)\,d\theta\right)\,d\phi \\
&= \int_{\rho=0}^{\rho=a}\pi\rho^3\,d\phi \\
&= \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=a} = \frac{a^4}{4}
\end{aligned}$$

The volume of the half ball is $2\pi a^3/3$ and so we get

$$\bar{z} = \frac{a^4/4}{2\pi a^3/3} = \frac{3}{8}a$$

The centre of mass is at $(0, 0, (3/8)a)$, or $3/8$ of the way along the central radius from the centre.

TO BE CHECKED

Richard M. Timoney (February 13, 2007)