Q. If H is cyclic, then $\phi(H)$ is cyclic.

Proof: If $H = \langle a \rangle$, then $\phi(H) = \langle \phi(a) \rangle$.

Given that H is cyclic subgroup of G.
then, there exist $a \in H$ such that

$$H = \langle a \rangle \quad \text{—— (1)}$$

Given that $\phi : G \to \overline{G}$ is a group homomorphism.
Let e & $\overline{e}$ be the identity elements of G & $\overline{G}$ respectively.

Now, $\phi(H) = \{ \phi(h) \mid h \in H \}$ —— (2)

We claim that $\phi(H) = \langle \phi(a) \rangle$.

Let $x \in \phi(H)$ be an arbitrary element.
$\because x \in \phi(H) \Rightarrow \exists\, h \in H$ such that

$$\phi(h) = x \quad \text{—— (3)}$$

$\because h \in H$ & $H = \langle a \rangle$

$\therefore h = a^k$ for some $k \in \mathbb{Z}$.

From eqn (3), $\phi(a^k) = x$

$\therefore (\ldots)^k$ [$\because \phi$ is a]

$\Rightarrow \left(\phi(a)\right)^k = x$ [$\because \phi$ is a homomorphism]

$\Rightarrow x \in \langle \phi(a) \rangle$.

$\therefore \phi(H) \subseteq \langle \phi(a) \rangle$ ——— (4)

$\because a \in H \Rightarrow \phi(a) \in \phi(H)$

$\Rightarrow \langle \phi(a) \rangle \subseteq \phi(H)$ ——— (5)

From eq^n (4) & (5), $\phi(H) = \langle \phi(a) \rangle$

$\therefore \phi(H)$ is cyclic.

Q. Show that $\phi : \mathbb{C}^* \to \mathbb{C}^*$ given by $\phi(x) = x^4$ is a homomorphism. Find all the elements that map to 2.

Sol^n// $\mathbb{C}^* \to$ Set of all complex numbers without zero.

$\mathbb{C}^*$ is a group under the operation 'multiplication'

TS// ϕ is a homomorphism.

(i) ϕ is well defined
let $x, y \in \mathbb{C}^*$
$x = y$
$\Rightarrow x^4 = y^4$

(ii) ϕ is O.P.
$\phi(xy) = (xy)^4$
$= x^4 y^4$

$x = y$

$\Rightarrow x^4 = y^4$

$\Rightarrow \phi(x) = \phi(y)$.

ϕ is well defined

$= x^4 y^4$

$= \phi(x)\,\phi(y)$

$\Rightarrow \phi$ is O.P.

Thus, ϕ is a homomorphism from $\mathbb{C}^*$ to $\mathbb{C}^*$.

$\{x \in \mathbb{C}^* \mid \phi(x) = 2\} = \phi^{-1}(2)$

Thm If $\phi(g) = g'$, then $\phi^{-1}(g') = g \ker\phi$

If $\phi(x) = x^4 = 2 \Rightarrow x^4 = 2 \Rightarrow x = 2^{1/4}$.

$\ker\phi = \{x \in \mathbb{C}^* \mid \phi(x) = 1\}$

$= \{x \in \mathbb{C}^* \mid x^4 = 1\}$

$= \{1, -1, i, -i\}$

$\phi(2^{1/4}) = 2$

$\Rightarrow g = 2^{1/4},\ g' = 2$

$\phi(g) = g'$

$\phi^{-1}(g') = g\ker\phi$

$\therefore \phi^{-1}(2) = 2^{1/4}\ker\phi = 2^{1/4}\{1, -1, i, -i\}$

$= \{2^{1/4}, -2^{1/4}, i2^{1/4}, -i2^{1/4}\}$

Thus, the set of all elements that map to 2

is $\{2^{1/4}, -2^{1/4}, i2^{1/4}, -i2^{1/4}\}$