Properties of External Direct Product :-

Thm 8.1 :- The order of an element of a direct product of finite no. of finite groups is the least common multiple of the orders of the components of the element.

Proof :- Let $G_1, G_2, \ldots, G_n$ be finite grps.

let $(g_1, g_2, \ldots, g_n) \in G_1 \oplus G_2 \oplus \cdots \oplus G_n$, then

$$|(g_1, g_2, \ldots, g_n)| = \text{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$$

Proof :- First we prove for $n = 2$.

let G_1, G_2 be two finite groups and let $(g_1, g_2) \in G_1 \oplus G_2$ be arbitrary.

Claim :- $|(g_1, g_2)| = \text{lcm}(|g_1|, |g_2|)$.

let $s = \text{lcm}(|g_1|, |g_2|)$ and $t = |(g_1, g_2)|$.

then $(g_1, g_2)^s = (g_1^s, g_2^s) = (e, e)$

$\Rightarrow$ t divides s. — (1)

Also, As $(g_1, g_2)^t = (e, e) = (g_1^t, g_2^t)$

$\Rightarrow$ $|g_1|$ and $|g_2|$ divides t

$\Rightarrow$ $\text{lcm}(|g_1|, |g_2|)$ divides t

$\Rightarrow$ s divides t — (2)

from (i) and (ii),

$s = t$.

$\Rightarrow |(g_1, g_2)| = \text{lcm}(|g_1|, |g_2|)$.

Now we consider the case of any natural no. n,

Let $G_1, G_2, \dots, G_n$ be finite gps and

Let $(g_1, g_2, \dots, g_n) \in G_1 \oplus G_2 \oplus \dots \oplus G_n$ be arb.

Claim:- $|(g_1, g_2, \dots, g_n)| = \text{lcm}(|g_1|, |g_2|, \dots |g_n|)$.

Let $s = \text{lcm}(|g_1|, |g_2|, \dots, |g_n|)$

$t = |(g_1, g_2, \dots, g_n)|$

then $(g_1, g_2, \dots, g_n)^s = (g_1^s, g_2^s, \dots, g_n^s)$

$= (e, e, \dots, e)$

$\Rightarrow$ t divides s. — (i)

Also, $(g_1, g_2, \dots, g_n)^t = (e, e, \dots, e)$

$= (g_1^t, g_2^t, \dots, g_n^t)$

$\Rightarrow |g_1|, |g_2| \dots \& |g_n|$ divides t.

$\Rightarrow \text{lcm}(|g_1|, \dots, |g_n|)$ divides t

$\Rightarrow$ s divides t — (ii)

from (i) & (ii)

$s = t$

Examples:-

(1) Determine the no. of elements of order 5 in $Z_{25} \oplus Z_5$.

$\Rightarrow$ Let $(a, b) \in Z_{25} \oplus Z_5$ s.t.

$$|(a,b)| = 5 = \text{lcm}(|a|, |b|).$$

As $\text{lcm}(|a|, |b|) = 5$, there are 3 possibilities

Case (1) If $|a| = 5$ and $|b| = 5$

Then there are 4 choices of a and 4 choices of b

$\Rightarrow$ We ~~have~~ get 16 elements of order 5.

Case (2) If $|a| = 5$ and $|b| = 1$.

Then there are 4 choices of a and 1 choice of b.

$\therefore$ We ~~have~~ get 4 elements of order 5.

Case (3) If $|a| = 1$ and $|b| = 5$

Then there are 1 choice of a and 4 choices of b.

$\therefore$ We get 4 elements of order 5.

Thus $Z_{25} \oplus Z_5$ have 24 elements of order 5.

Ex② Determine the no. of elements of order 5 in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$.

$\Rightarrow$ Let $(a,b) \in \mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$ s.t.

$$|(a,b)| = 5 = \operatorname{lcm}(|a|, |b|).$$

Case 1, $|a| = 5$ and $|b| = 1$

$\Rightarrow$ a has 4 choices and b has 1 choice.

$\therefore$ We get 4 elements of order 5.

Case 2: $|a| = 5$ and $|b| = 5$.

$\Rightarrow$ a has 4 choices and b has 4 choices.

$\therefore$ We get 16 elements of order 5.

Case 3 $|a| = 1$ and $|b| = 5$.

$\Rightarrow$ a has 1 choice and b has 4 choices

$\therefore$ We get 4 elements of order 5.

$\Rightarrow$ $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$ has 24 elements of order 5.

Ex③ Determine the no. of cyclic subgps of order 10 in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$.

$\Rightarrow$ First we calculate the no. of elements of order 10 in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$. Let $(a,b) \in \mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$

$|a| = 10$ & $|b| = 1$, $|a| = 10$ & $|b| = 5$, $|a| = 2$ & $|b| = 5$

We have these 3 cases for $|(a,b)| = 10$.

Case 1 $|a| = 10$, $|b| = 1$

then a has 4 choices and b has 1 choice

$\therefore$ We get 4 elements of order 10.

Case 2 $|a| = 10$ & $|b| = 5$.

a has 4 choices and b has 4 choices.

$\therefore$ We get 16 elements of order 10.

Case 3 $|a| = 2$ & $|b| = 5$

a has 1 choice and b has 4 choices.

$\therefore$ We get 4 elements of order 10.

$\Rightarrow$ $Z_{100} \oplus Z_{25}$ has 24 elements of order 10.

Now every cyclic subgp of order 10 has 4 elements of order 10 and no two subgps can have a common element of order 10.

Therefore, there are $24/4 = 6$ cyclic subgps of order 10.

Ex 4 How many subgps of order 4 does $Z_4 \oplus Z_2$ have.

Soln:- First we calculate cyclic subgps of order 4. for that we need to calculate no. of elements of order 4.

let $(a, b) \in Z_4 \oplus Z_2$ s.t.

$$|(a,b)| = 4.$$

Case 1 $|a| = 4$, $|b| = 1$

a has 4 choices, b has 1 choice

$\rightarrow$ we get 4 elements.

Case 2 $|a| = 4$, $|b| = 2$

a has 4 choices, b has 1 choice

$\rightarrow$ we get 4 elements.

$\therefore$ We have 8 elements of order 4 in $Z_4 \oplus Z_2$

And every cyclic gp of order 4 has 2 elements of order 4.

$\therefore$ We have $\frac{8}{2} = 4$ cyclic subgps of order 4 in $Z_4 \oplus Z_2$.

Non-cyclic subgps:-

Any subgp of order 4 which is not cyclic has order 2 for every non-identity element and $Z_4 \oplus Z_2$ has 3 elements of order 2, namely $(2,0)$, $(2,1)$ & $(0,1)$

$\therefore$ $Z_4 \oplus Z_2$ has only 1 non-cyclic subgp of order 4.

$\therefore$ $Z_4 \oplus Z_2$ has 5 subgp of order 4.

Thm 8.2 Let G and H be finite cyclic gp. Then $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

Proof:- Let $|G| = m$ and $|H| = n$

then $|G \oplus H| = mn$.

Let $G \oplus H$ is cyclic

Claim:- m and n are relatively prime.

Suppose $\gcd(m,n) = t \neq 1$.

then t/m and t/n.

Let $g \in G$ s.t. $|g| = m$ & $h \in H$ s.t. $|h| = n$.

then $\langle (g^{m/t}, e) \rangle$ and $\langle (e, h^{n/t}) \rangle$ are distinct subgp of order in t in $G \oplus H$ (which is cyclic)

This is a contradiction.

$\therefore$ $t = 1$ $\Rightarrow$ $\gcd(m,n) = 1$.

Conversely,

let $\gcd(m,n) = 1$

Claim:- $G \oplus H$ is cyclic

Let $g \in G$ s.t. $|g| = m$ & $h \in H$ s.t. $|h| = n$

then $|(g,h)| = \text{lcm}(|g|,|h|) = \text{lcm}(m,n)$
$= mn$.

$\therefore \langle (g,h) \rangle = G \oplus H$

$\Rightarrow G \oplus H$ is cyclic ▪.

Corollary!- An external direct product $G_1 \oplus G_2 \oplus \dots \oplus G_n$ of a finite number of finite cyclic grp is cyclic if and only if $|G_i|$ and $|G_j|$ are relatively prime when $i \neq j$.

Proof:- We can prove it by Induction.

For $n=2$, we have shown (thm 8.2)

Now let us assume that the result is true for $n = k-1$

i.e. $G_1 \oplus \dots \oplus G_{k-1}$ is cyclic iff $|G_i|$ & $|G_j|$ are relatively prime.

Now we prove the result for $n = k$.

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Consider $G_1 \oplus G_2 \oplus \dots \oplus G_k$ and let $|G_i|$ & $|G_j|$ are relatively prime for $i \neq j$.

Let $G = G_1 \oplus \dots \oplus G_{k-1}$

then $G_1 \oplus \dots \oplus G_k = G \oplus G_k$

and $|G|$ and $|G_k|$ are relatively prime

$\therefore$ $G \oplus G_k$ is cyclic [$\because$ theorem 8.2]

$\Rightarrow$ $G_1 \oplus \dots \oplus G_k$ is cyclic.

Ilły we can do converse.

Corollary 2:- Let $m = n_1 n_2 \dots n_k$. Then Z_m is isomorphic to $Z_{n_1} \oplus Z_{n_2} \oplus \dots \oplus Z_{n_k}$ if and only if n_i and n_j are relatively prime.

Ex:- $Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_5 \approx Z_2 \oplus Z_6 \oplus Z_5$

$\approx Z_2 \oplus Z_2 \oplus Z_{15}$

$\approx Z_2 \oplus Z_3 \oplus Z_{10}$

$\approx Z_6 \oplus Z_{10}$