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Vector space

Linear Transformation:

Null space, Range space, nullity, rank,

Dimension theorem

If $T: V \to W$ is linear, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

Thm: Let V and W be vector spaces over field F and suppose that the set $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. For $w_1, w_2, \ldots, w_n$ in W, there exists exactly one linear transformation $T: V \to W$ such that

$$T(v_i) = w_i \text{ for } i = 1, 2, \ldots, n.$$

Proof:

TS T is linear and T is unique

TS: T is linear and T is unique such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

let $x \in V$

$\ni$ x can we written as

$$x = \sum_{i=1}^{n} a_i v_i, \text{ where } a_i \in F, 1 \leq i \leq n.$$

[$\{v_1, v_2, \ldots, v_n\}$ is a basis for V]

Define $T: V \to W$ such that

$$\ni T(x) = \sum_{i=1}^{n} a_i w_i \quad \text{—— (1)}$$

First we will show that, T is linear.

let $u, v \in V$ and $k \in F$.

Then u and v can be written as

$$u = \sum_{i=1}^{n} b_i v_i \text{ and } v = \sum_{i=1}^{n} c_i v_i \quad \text{—— (2)}$$

where b_i's & c_i's $\in F$, $1 \le i \le n$.

Now, $$T(u+v) = T\left(\sum_{i=1}^{n} b_i v_i + \sum_{i=1}^{n} c_i v_i\right)$$

$$= T\left(\sum_{i=1}^{n} (b_i + c_i) v_i\right)$$

$$= \sum_{i=1}^{n} (b_i + c_i)\, T(v_i)$$

$$= \sum_{i=1}^{n} b_i T(v_i) + \sum_{i=1}^{n} c_i T(v_i)$$

$$= \sum_{i=1}^{n} T(b_i v_i) + \sum_{i=1}^{n} T(c_i v_i)$$

$$= T\left(\sum_{i=1}^{n} b_i v_i\right) + T\left(\sum_{i=1}^{n} c_i v_i\right)$$

$$= T(u) + T(v)$$

$\therefore$ $T(u+v) = T(u) + T(v)$. ——— (3)

Now, $T(ku) = T\left(k\sum_{i=1}^{n} b_i v_i\right)$

$$= k\,T\left(\sum_{i=1}^{n} b_i v_i\right)$$

$$= k\,T(u)$$

$\therefore T(ku) = kT(u)$ —— 4

from eqn 3 & 4, we have

$T(u+v) = T(u) + T(v)$

$T(ku) = kT(u)$

$\therefore T$ is linear.

Now T is defined as

$T\left(\sum_{i=1}^{n} a_i v_i\right) = \sum_{i=1} a_i w_i$ —— 5

(using eqn 1).