Non - Homogeneous Linear Differential equations with Constant coefficients :-

The Differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x) \quad \text{— (I)}$$

is called non- homogeneous Linear differential equation with constant coefficient, where $a_0, a_1, -- a_n$ are constant and $f(x) \neq 0$.

The soln of equation (1) is given as

$$y(x) = \underbrace{y_c}_{\text{Complementary function}} + \underbrace{y_p}_{\text{Particular Integral}} \quad \text{— (2)}$$

where y_c is called complementary function which is a general soln of corresponding homogenous eqn of (1),

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \quad \text{— (2)}$$

and y_p is called particular integral which is a solution corresponding to $F(x)$,

In the last section we have learn how to find the general solution or complementary function y_c of corresponding homogeneous eq^n of (1).
In this section, we will learn, how to find particular integral y_p or sol^n corresponding to $f(x)$.

Now, we denote $D = \frac{d}{dx}$, then eq^n (1) becomes

$$a_0 D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = F(x)$$

or

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)\, y = F(x)$$

or

$$\boxed{f(D)\, y = F(x)}$$

This is symbolic notation of eq^n (1),

and the particular Integral is

$$\boxed{y_p = \frac{1}{f(D)} F(x).}$$

where $f(D)$ is an operator and $\frac{1}{f(D)}$ is the inverse of operator $f(D)$,

Example :- $\frac{d^2y}{dx^2} + (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y = F(x)$

or

$$[D^2 - (\alpha+\beta) D + \alpha\beta]\, y = F(x) \quad \left[\text{as } D = \frac{d}{dx}\right]$$

then $f(D) = D^2 - (\alpha+\beta)D + \alpha\beta$

and

particular integral of the given eqn is

$$= \frac{1}{D^2-(\alpha+\beta)D+\alpha\beta} F(x)$$

or

$$\boxed{y_p = \frac{1}{(D-\alpha)(D-\beta)} F(x)}.$$

Here $\frac{1}{f(D)} = \frac{1}{(D-\alpha)(D-\beta)}$.

Now, we will discuss methods to find P.I. (y_p).

Methods of finding the particular integral :-

(a) If operator $f(D)$ can be written in factors like

$$f(D) = (D-m_1)(D-m_2)\cdots(D-m_n)$$

the P.I of $f(D)y = F(x)$ will be

$$\frac{1}{(D-m_1)} \cdot \frac{1}{(D-m_2)} \cdots \frac{1}{(D-m_n)} F(x)$$

and operate from right, we obtained

$$P.I. = \frac{1}{(D-m_1)} \cdot \frac{1}{(D-m_2)} \cdots \frac{1}{(D-m_{n-1})} e^{m_n x} \int e^{-m_n x} F(x)\,dx$$

~~Contine~~ on operating with second and remaining factor in succession, taking them from right to left, we obtained final value of

$$P.I. = e^{m_1 x} \int e^{-m_1 x} \int \cdots \int e^{-m_n x} f(x) (dx)^n.$$

To understand above method, we will take very simple D.E.

$$\frac{dy}{dx} - ay = f(x)$$

or

$$(D-a)y = f(x)$$

The particular Integral of above eqn is

$$P.I. = \frac{1}{(D-a)} f(x)$$

or

$$\boxed{P.I. = e^{ax} \int e^{-ax} f(x)\, dx}$$

Example :- Solve. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$ —— (1)

Corresponding homogeneous eq^n of (1) is

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \quad \text{—— (2)}$$

Auxiliary eq^n of (2) is

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

Hence complementary function

$$\boxed{y_c = C_1 e^{2x} + C_2 e^{3x}}$$

Rewrite eqn (1)

$$(D^2 - 5D + 6)y = e^{4x}$$

or

$$(D-2)(D-3)y = e^{4x}$$

Particular Integral of above eqn is

$$y_p = \frac{1}{(D-2)(D-3)} e^{4x}$$

or

$$y_p = \frac{1}{(D-2)} e^{3x} \int e^{-3x} \cdot e^{4x} \, dx$$

$$= \frac{1}{(D-2)} \cdot e^{3x} \int e^{x} \, dx$$

$$= \frac{1}{(D-2)} e^{4x}$$

$$y_p = e^{2x} \int e^{-2x} e^{4x} \, dx$$

$$y_p = e^{2x} \int e^{2x} \, dx = e^{2x} \times \frac{1}{2} e^{2x}$$

$$\boxed{y_p = \frac{1}{2} e^{4x}}$$

and hence general solⁿ of eqn (1) is

$$\boxed{y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{2} e^{4x}}$$

Exercise :- (1) $\frac{d^2y}{dx^2} - y = 2 + 5x$

Exercise : (2) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{2x}$