Laplace Transform

The Laplace transform of a function F(t) is denoted by $L[F(t)]$ or $f(s)$ and defined as

$$\boxed{f(s) = L[F(t)] = \lim_{a \to \infty} \int_0^a e^{-st} F(t)\,dt = \int_0^\infty e^{-st} F(t)\,dt}, \quad t \geq 0$$

Here L operator is known as Laplace transformation operator.

$s =$ real or complex no. generally taken +ve.

Note: — It helps in solving the diff. eqns. with boundaries values w/out finding general solⁿ & the values of arbitrary constts.

Note: $L[F(t)]$ exists only if

(i) F(t) should be an arbitrary piecewise conti function in every finite interval and that $F(t)=0 \; \forall \; t$.

(ii) F(t) should be of exponential order.

Properties

(1) Linearity Property;

$$L[a_1 F_1(t) + a_2 F_2(t)] = a_1 L[F_1(t)] + a_2 L[F_2(t)]$$

where a_1 & a_2 are constts.

Proof:

LHS

$$L[a_1 F_1(t) + a_2 F_2(t)] = a_1 \int_0^\infty e^{-st} F_1(t)\,dt + a_2 \int_0^\infty e^{-st} F_2(t)\,dt$$

$$= a_1 L[F_1(t)] + a_2 L[F_2(t)]$$

or Generalised form can be written as

$$\boxed{L\left[\sum_{m=1}^{n} a_m F_m(t)\right] = \sum_{m=1}^{n} a_m L[F_m(t)]}$$

(2) Change of Scale Property;

If $f(s)$ is the L.T of $F(t)$ then L.T of $F(at)$ is $\frac{1}{a} f\left(\frac{s}{a}\right)$

Proof:

$$f(s) = L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt \quad \text{---- (1)}$$

$$\therefore L[F(at)] = \int_0^\infty e^{-st} F(at)\,dt \quad \text{---- (2)}$$

put $at = u \Rightarrow dt = \frac{du}{a}$ in (2)

$$\therefore L[F(at)] = \int_0^\infty e^{-\left(\frac{s}{a}\right)u} F(u)\frac{du}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)u} F(u)\,du$$

$$\boxed{L[F(at)] = \frac{1}{a} f\left(\frac{s}{a}\right)}$$

(3) First translation or shifting property:

if $f(s)$ is the L.T. of $F(t)$ then the L.T. of $e^{at}F(t)$ will be $f(s-a)$ ie $\boxed{L[e^{at}F(t)] = f(s-a)}$

Proof: we know that $L[F(t)] = \int_0^\infty e^{-st} F(t)dt = f(s)$

$$\therefore L[e^{at}F(t)] = \int_0^\infty e^{-st} e^{at} F(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} F(t)dt$$

$$= \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^\infty = f(s-a)$$

$$\therefore \boxed{L[e^{at}F(t)] = f(s-a)}$$

(4) Second translation property or Heaviside Shifting Theorem.

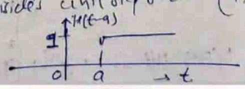
if $L[F(t)] = f(s)$ and $G(t) = \begin{cases} 0 ; & 0<t<a \\ F(t-a) ; & t>a \end{cases}$

then $\boxed{L[G(t)] = e^{-as}f(s)}$

we can express $L[F(t-a)H(t-a)] = e^{-sa} f(s)$

where $H(t-a) = \begin{cases} 0 ; & t<a \\ 1 , & t \geq a \end{cases}$

called Heaviside's unit step function $(H_a(t))$



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Proof: # $\boxed{L[F(t-a)H(t-a)] = e^{-as}f(s)}$ Second Shifting theorem or Heaviside's "

$$L[F(t-a)H(t-a)] = \int_0^\infty e^{-st}[F(t-a)\cdot H(t-a)]\,dt$$

$$= \underbrace{\int_0^a e^{-st}F(t-a)\cdot 0\,dt}_{=0} + \int_a^\infty e^{-st}F(t-a)\cdot 1\,dt$$

$$= \int_0^\infty e^{-st}F(t-a)\,dt$$

put $t-a = u$, $dt = du$

$$= \int_0^\infty e^{-s(u+a)}F(u)\,du$$

$$= \int_0^\infty e^{-as}\cdot e^{-su}F(u)\,du$$

$$= e^{-as}f(s)$$ Hence proved

$\boxed{L[F(t)H(t-a)] = e^{-as}L[F(t+a)]}$

Proof:- $L[F(t)H(t-a)] = \int_0^\infty e^{-st}[F(t)H(t-a)]dt$

$$= \underbrace{\int_0^a e^{-st}F(t)\cdot 0\,dt}_{=0} + \int_a^\infty e^{-st}F(t)\cdot 1\,dt \quad \text{--(i)}$$

Now put $(t-a) = u \Rightarrow dt = du$

(i) becomes

$$= \int_0^\infty e^{-s(a+u)}F(a+u)\,du$$

$$= e^{-as}\int_0^\infty e^{-su}F(a+u)du$$

$\boxed{L[F(t)H(t-a)] = e^{-as}L[F(a+u)]}$ Hence proved.

Problems

① find L.T. of 1, K, t, Kt, e^{at} and e^{-at}

Sol^n:

$$F(t)=1$$

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt$$

$$\Rightarrow L(1) = \int_0^\infty e^{-st}\,dt = -\frac{1}{s} e^{-st} \Big]_0^\infty = \frac{1}{s} \quad \underline{\text{Ans}}$$

$$\Rightarrow L[K] = \int_0^\infty e^{-st} K\,dt = K\left(\frac{e^{-st}}{-s}\right)_0^\infty = \frac{K}{s} \quad \underline{\text{Ans.}}$$

$$\Rightarrow L[t] = \int_0^\infty \underset{II}{e^{-st}}\,\underset{I}{t}\,dt$$

$$= \left(t\,\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty 1 \cdot \frac{e^{-st}}{-s}\,dt$$

$$= 0 + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s^2}$$

$$\Rightarrow L[Kt] = \int_0^\infty e^{-st} Kt\,dt = K\int_0^\infty e^{-st}\,t\,dt$$

$$= \frac{K}{s^2} \quad \text{Ans.}$$

$$\Rightarrow L[e^{at}] = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left(\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a} \quad \text{if } \operatorname{Re} s > a \quad \underline{\text{Ans}}$$

$$\Rightarrow L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\,dt = \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a} \quad \underline{\text{Ans}}$$

[redacted]

find L.T. of functions (i) $e^{at}\cos\omega t$ (ii) $e^{at}\sin\omega t$

Sol^n: (i) $e^{at}\cos\omega t$

$$L[e^{at}\cos\omega t] = \int_0^\infty e^{-st}e^{at}\cos\omega t \, dt$$

$$= \int_0^\infty e^{-st}e^{at}\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-a-i\omega)t} + e^{-(s-a+i\omega)t}\right)dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a-i\omega)t}}{-(s-a-i\omega)} + \frac{e^{-(s-a+i\omega)t}}{-(s-a+i\omega)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-a-i\omega} + \frac{1}{s-a+i\omega}\right]$$

$$= \frac{s-a}{(s-a)^2+\omega^2} \quad \text{Ans} \quad s>a, >\omega$$

II method

$$L[\cos\omega t] = \int_0^\infty e^{-st}\cos\omega t \, dt$$

$$= \int_0^\infty e^{-st}\cdot\left[\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right]dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-i\omega)t} + e^{-(s+i\omega)t}\right]dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-i\omega)t}}{-(s-i\omega)} + \frac{e^{-(s+i\omega)t}}{-(s+i\omega)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right]$$

$$= \frac{1}{2}\cdot\frac{s}{s^2+\omega^2}$$

we know

$$L[e^{at}f(t)] = f(s-a) = \frac{1}{2}\frac{(s-a)}{(s-a)^2+\omega^2} \quad \text{Ans}$$

(ii) $e^{at}\sin\omega t$,

$$L[e^{at}\sin\omega t] = \int_0^\infty e^{-st}e^{at}\sin\omega t \, dt$$

$$= \int_0^\infty e^{-st}e^{at}\frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)dt$$

$$= \frac{1}{2i}\int_0^\infty \left[e^{-(s-a-i\omega)t} - e^{-(s-a+i\omega)t}\right]dt$$

$$= \frac{1}{2i}\left[\frac{1}{s-a-i\omega} - \frac{1}{s-a+i\omega}\right]_0^\infty$$

$$= \frac{1}{2i}\left[\frac{2i\omega}{(s-a)^2+\omega^2}\right] = \frac{\omega}{(s-a)^2+\omega^2}, \quad s>a, >\omega$$

Ans

find L.T. of $\frac{e^{at}-1}{a}$

Sol^n: -

$$L\left[\frac{e^{at}-1}{a}\right] = \int_0^\infty e^{-st}\left(\frac{e^{at}-1}{a}\right)dt$$

$$= \frac{1}{a}\left[\int_0^\infty e^{-(s-a)t}dt - \int_0^\infty e^{-st}dt\right]$$

$$= \frac{1}{a}\left(\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-st}}{-s}\right)_0^\infty$$

$$= \frac{1}{a}\left(\frac{1}{s-a} - \frac{1}{s}\right)$$

$$= \frac{1}{s(s-a)} \quad \text{Ans}$$