Weak Law of Large Numbers and Central Limit Theorem

8.1 Markov's Inequality

If X is a non negative random variable, then for any $a > 0$, $P[X \geq a] \leq \frac{E[X]}{a}$

Proof: Let X be a continuous random variable having density $f(x)$

$$E[X] = \int_0^\infty x f(x)dx = \int_0^a x f(x)dx + \int_a^\infty x f(x)dx$$

$$\geq \int_a^\infty x f(x)dx$$

$$\geq \int_a^\infty a f(a)dx$$

$$= a \int_a^\infty f(x)dx$$

$$= a P[X \geq a]$$

$$\therefore \frac{E[X]}{a} \geq P[X \geq a]$$

Hence $P[X \geq a] \leq \frac{E[X]}{a}$

8.2 Chebyshev's Inequality

Let X be a random variable with mean μ and variance σ^2. Then for any $k > 0$

$$P[|X-\mu| \geq k] \leq \frac{\sigma^2}{k^2}$$

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Proof: Since $(X-\mu)^2$ is a non negative random variable, then by Markov's inequality we have

$$P[(X-\mu)^2 \geq k^2] \leq \frac{E[(X-\mu)^2]}{k^2}$$

$$\Rightarrow P[|X-\mu| \geq k] \leq \frac{\sigma^2}{k^2} \quad \left[\because x^2 \geq a^2 \text{ iff } |x| \geq a \text{ for any } a > 0\right]$$

Hence proved.

Remark 1. Another form of Chebyshev's Inequality is

$$P[|X-\mu| < k] \geq 1 - \frac{\sigma^2}{k^2}$$

This follows as

$$P[|X-\mu| < k] + P[|X-\mu| \geq k] = 1$$

$$\Rightarrow P[|X-\mu| < k] = 1 - P[|X-\mu| \geq k]$$

$$\geq 1 - \frac{\sigma^2}{k^2} \quad [\text{Using above form}]$$

Remark 2. The Chebyshev's inequality gives a bound for the probability of a event given in terms of a random variable and its mean and variable. Also this bound is independent of distribution of X

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Example 1 → Let X has probability density

$$f(x) = \begin{cases} 630x^4(1-x)^4 & \text{for } 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the probability that it will take on a value within two standard deviations of mean and compare this probability with the lower bound provided by Chebyshev's Inequality

Sol: $$f(x) = \begin{cases} 630x^4(1-x)^4 & ; 0<x<1 \\ 0 & ; \text{elsewhere} \end{cases}$$

The mean (μ) of R.V X is given by

$$\mu = E(X) = \int_0^1 x f(x)\,dx$$

$$= \int_0^1 630x^5(1-x)^4\,dx$$

$$= 630 \int_0^1 630x^5[1+x^2-2x][1+x^2-2x]$$

$$= 630 \int_0^1 x^5(x^4-4x^3+6x^2-4x+1)$$

$$= 630\cdot\left[\frac{x^{10}}{10} - 4\frac{x^9}{9} + 6\frac{x^8}{8} - 4\frac{x^7}{7} + \frac{x^6}{6}\right]$$

$$= 630\left[\frac{1261-2520}{1260}\right] = 1/2$$

Now $$E(X^2) = \int_0^1 x^2 f(x)\,dx$$

$$= \int_0^1 630x^6(1-x)^4\,dx$$

$$= 630\left[\int_0^1 x^6(x^4-4x^3+6x^2-4x+1)\right]$$

$$= 630\left[\frac{x^{11}}{11} - 4\frac{x^{10}}{10} + 6\frac{x^9}{9} - 4\frac{x^8}{8} + \frac{x^7}{7}\right]$$

$$\frac{3}{11} = \quad = 630\left(\frac{1}{11} - \frac{2}{5} - \frac{2}{3} - \frac{1}{2} + \frac{1}{7}\right) = \frac{630(2080-2079)}{2310}$$

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Hence Variance $= \sigma^2 = E(X^2) - [E(X)]^2$

$$= \frac{3}{11} - \frac{1}{4} = \frac{1}{44}$$

$$\therefore \sigma = \sqrt{\text{variance}} = \sqrt{\frac{1}{144}} \simeq 0.15$$

Now the probability that X will take on a value within two standard deviations of the mean is equal to the probability that it will take on a value between 0.20 and 0.80 is

$$P[0.20 < X < 0.80] = \int_{0.20}^{0.80} 630\, x^4 (1-x)^4\, dx$$

put $x = \cos^2\theta$
$dx =$

$$= 630 \int_{0.20}^{0.80} x^4 [x^4 - 4x^3 + 6x^2 - 4x + 1]\, dx$$

$$= 630 \int_{0.20}^{0.80} (x^8 - 4x^7 + 6x^6 - 4x^5 + x^4)\, dx$$

$$= 630 \left[\frac{x^9}{9} - 4\frac{x^8}{8} + \frac{6x^7}{7} - \frac{4x^6}{6} + \frac{x^5}{5}\right]_{0.20}^{0.80}$$

$$= 0.96 \quad -(1)$$

Now Using Chebyshev's Enequality

$$P[0.20 < X < 0.80] = P[-0.30 < X - \mu < 0.30]$$

$$= P[|X - \mu| < 0.30]$$

$$= P[|X - \mu| < 2\sigma]$$

$$\geq 1 - \frac{1}{4} = 0.75 \quad -(2)$$

It is observed that the statement (1) that the probability is '0.96' is a much stronger statement than (2) which says that probability is atleast '0.75'

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Example 2. Suppose the number of items produced in a factory during a week is a random variable with mean 500. Then

(i) What can be said about the probability that this week's production will be atleast 1000?

(ii) If the variance of week's production is known to equal 100, then what can be said about the probability that this week's production will be between 400 and 600?

Sol: Let X be the number of items produced in a factory during a week

Given $E(X) = 500$

(i) By Markov's inequality

$$P[X \geq 1000] \leq \frac{E[X]}{1000} = \frac{500}{1000} = \frac{1}{2}$$

(ii) As $\sigma^2 = 100$ & $E(X) = \mu = 500$

$\therefore$ By Chebyshev's inequality

$$P[|X-500| \geq 100] \leq \frac{\sigma^2}{(100)^2} = \frac{1}{100}$$

$$\Rightarrow P[|X-500| < 100] \geq 1 - \frac{1}{100} = \frac{99}{100} = 0.99$$

$$\Rightarrow P[400 < X < 600] \geq 0.99$$

Therefore the probability that this week's production will be between 400 and 600 is atleast 0.99

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Example: A die is thrown 360 times, show that the number of sixes lies between 550 and 650 is atleast $\frac{4}{5}$

Sol: Let X is the no. of sixes, then X follows binomial distribution

ie $X \sim b(n, \theta)$

where $n = 360$, $\theta = \frac{1}{6}$, $1-\theta = \frac{5}{6}$

As $E(X) = n\theta = 60$ and

$V[X] = n\theta(1-\theta) = 50$

$$\therefore P[550 < X < 650] = P[-50 < X - \mu < 50]$$

$$= P[|X-\mu| < 50]$$

$$\geq 1 - \frac{Var(X)}{(50)^2}$$

$$= 1 - \frac{50}{2500} \quad \text{[Using Chebyshev's inequality]}$$

$$= 1 - \frac{1}{5} = \frac{4}{5}$$

Hence $P[550 < X < 650] \geq \frac{4}{5}$ proved

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