Linear Differential Equations with Constant Coefficients :—

The general form of L.D.E. with constants coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = F(x) \quad —(1)$$

where $a_0, a_1, a_2, \ldots a_{n-1}, a_n$ are real constants.

If $F(x) = 0$, then eq^n (1) becomes

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = 0 \quad —(2)$$

Equation (2) is called homogeneous Linear Differential equation of nth order.

Now, we will learn how to find solution of the homogeneous Linear Differential equations with Constant Coefficient.

Let $y = e^{mx}$ be the sol^n of eq^n (2),

then $\frac{dy}{dx} = m e^{mx}$, $\frac{d^2y}{dx^2} = m^2 e^{mx}$, $\cdots \rightarrow \frac{d^{n-1}y}{dx^{n-1}} = m^{n-1} e^{mx}$

& $\frac{d^n y}{dx^n} = m^n e^{mx}$

Putting the value of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}$ & $\frac{d^ny}{dx^n}$ in eqn (2), we get

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

or

$$(a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n) e^{mx} = 0$$

or

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0 \quad \text{--- (3)}$$

The eqn (3) is called the auxiliary Equation or characteristic eqn of the given eqn (2).

Hence, to solve eqn (2), we write the auxiliary equation (3) and solve it for m. The solution of eqn (2) ~~depend~~ depend on type of roots (real & distinct, real & repeated or complex). We will discuss all these cases.

Case I :- If roots of auxiliary are the n distinct real numbers

$$m_1, m_2, \ldots, m_n.$$

then general solution of eqn (2) is

$$\boxed{y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}}$$

and $e^{m_1 x}, e^{m_2 x}, \ldots, e^{m_n x}$ are n distinct

solutions of eqⁿ (2).

<u>Example</u> :– Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ — (I)

The auxiliary equation is

$$m^2 - 3m + 2 = 0.$$

hence

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

The roots are real and distinct. Thus e^x & e^{2x} are solⁿ of (I).

And general solⁿ of (I) is

$$\boxed{y(x) = c_1 e^x + c_2 e^{2x}}$$

<u>Case 2</u> :- <u>Repeated Real roots</u> :-

If the auxiliary equation (3) has the real root m occurring k times, and the remaining roots of the auxiliary equation (3) are the distinct real numbers $m_{k+1}, \ldots, m_n$, then the general solⁿ of (2).

$$y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{mx} + c_{k+1} e^{m_{k+1} x} + \cdots + c_n e^{m_n x}$$

Example :- Find the general soln of

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0 \quad \text{--- (I)}$$

The auxiliary equation of (I) is

$$m^3 - 4m^2 - 3m + 18 = 0$$

$$(m-3)^2(m+2) = 0$$

$$m = 3, 3, -2.$$

Then general solution is

$$\boxed{y(x) = (C_1 + x C_2) e^{3x} + C_3 e^{-2x}}$$

2

Case 3: Conjugate Complex Roots :-

If auxiliary equation has complex roots $a \pm bi$, then soln is

~~$y = C_1 \sin$~~

$$\boxed{y = e^{ax}(C_1 \sin bx + C_2 \cos bx)}$$

2

Example :- Find the general soln of

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \quad \text{--- (I)}$$

The auxiliary equation is

$$m^2 - 6m + 25 = 0$$

roots of this equation are $m = 3 \pm 4i$.

Here roots are the conjugate complex numbers $a \pm bi$, where $a = 3$, $b = 4$.

The general solution may be written

$$\boxed{y = e^{3x}(C_1 \sin 4x + C_2 \cos 4x)}$$

Initial Value Problem :-

Example :- Solve IVP

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \quad —(1)$$

$$y(0) = -3$$

$$y'(0) = -1$$

Auxiliary eq^n of (1) is

$$m^2 - 6m + 25 = 0$$

then, $m = 3 \pm 4i$

The general sol^n of (1) is

$$y = e^{3x}(C_1 \sin 4x + C_2 \cos 4x) \quad —(2)$$

& $y' = e^{3x}(4C_1 \cos 4x - 4C_2 \sin 4x) + 3e^{3x}(C_1 \sin 4x + C_2 \cos 4x$

Now, applying initial condition $y(0) = -3$ to eq^n (2), we get

$$-3 = e^0 (C_1 \sin 0 + C_2 \cos 0)$$

$$-3 = 1 \times (C_1 \times 0 + C_2 \times 1)$$

$$\boxed{C_2 = -3}$$

& applying initial condition $y'(0) = -3$

$$-1 = e^0 [(3C_1 - 4C_2) \sin 0 + (4C_1 + 3C_2) \cos 0]$$

{ as $y'(x) = e^{3x}[(3C_1 - 4C_2) \sin x + (4C_1 + 3C_2) \cos x]$ }

then, we get

$$\boxed{4C_1 + 3C_2 = -1}$$

& we have $C_2 = -3$, $\Rightarrow \boxed{C_1 = 2}$

therefore sol^n of (1) is

$$\boxed{y(x) = e^{3x}(2 \sin 4x - 3 \cos 4x)}$$