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Chapter 7. Q. 26

Suppose that G is a group with more than one element and G has no proper, nontrivial subgroups. Prove that |G| is prime. (Do not assume at the outset that G is finite.)

Sol^n

let G be infinite.

and $x \in G$. and $x \neq e$.

Assume that $|x| = \infty$,

then $\langle x \rangle \subseteq G$

$\langle x \rangle$ is subgroup of G. $\Rightarrow G = \langle x \rangle$

$\because |x| = \infty$, $\therefore x^k \neq (x^2)^m$ for any odd k.

$\therefore \langle x^2 \rangle$ is a subgroup of $\langle x \rangle$.

$\therefore \langle x^2 \rangle$ is a proper subgroup of G.

$\longrightarrow\!\leftarrow$

$\therefore |x| \neq \infty$

$\therefore$ All elements in G have finite order.

Now, If x has finite order,

then $\langle x \rangle$ is finite. but $G = \langle x \rangle$

$\longrightarrow\!\leftarrow$

$\therefore$ G can not be infinite.

~~Case-II:~~ G is finite, ~~let~~

let $e \neq x$ & $x \in G$

then ~~G~~ $\langle x \rangle$ is a subgroup of G
$\Rightarrow G = \langle x \rangle$ ($\because G$ has no nontrivial subgroup)

$\Rightarrow G$ is cyclic.

~~for any~~ let $|G| = |\langle x \rangle| = n$

for any positive divisor k of n, the group $\langle x \rangle$ has subgroup $\langle x^{n/k} \rangle$.

If k is the divisor of n, then $\langle x^{n/k} \rangle$ is the proper subgroup of $\langle x \rangle$.

$\longrightarrow\longleftarrow$

$\therefore$ n does not have any ~~divis~~ divisor other than 1 and n

$\Rightarrow n$ is prime.

$\Rightarrow |G|$ is prime.

Q. Every subgroup of index 2 is normal in a group.

Sol^n // let G be a group and H be a subgroup of G with index 2

$\because [G:H] = 2 \Rightarrow$ no. of distinct left (or right)

$\because [G:H] = 2 \Rightarrow$ no. of distinct left (or right) cosets is two

G is the union of all left (or right) cosets.

$G = eH \cup aH$ and $G = He \cup Ha$

$\Rightarrow G = H \cup aH$ and $G = H \cup Ha$ —— (1)

where $a \in G$

$\boxed{H \text{ is normal in } G \text{ if } aH = Ha}$

Now, let $a \in G$ then $a \in H$ or $a \notin H$

~~Case-I:~~ When $a \in H$

$aH = H = Ha$

$\Rightarrow aH = Ha \Rightarrow H \triangleleft G$.

$aH = H \Leftrightarrow a \in H$

$Ha = H \Leftrightarrow a \in H$

Case-II: When $a \notin H$.

from eqn (1), $G = H \cup aH$.

$\Rightarrow aH = G \setminus H$ —— (2)

Again from eqn (1), $G = H \cup Ha$

$\Rightarrow Ha = G \setminus H$ —— (3)

from eqn (2) & (3), $aH = Ha$

$\Rightarrow H \triangleleft G$

$\therefore$ every subgroup of index 2 is normal in group 5.

~~Chapter 9~~ 1 to 4, 6 to 18, 20, 21, 27, 36 to 45, 47, 49 to last question.