# Complex Analysis.
## Chapter - 5
## Series

## Sec 55. Convergence of Sequences

**Defn:** An infinite sequence $z_1, z_2, \ldots, z_n$ of complex numbers has a limit $z$, if for each

$\varepsilon > 0$, $\exists$ a +ve integer '$n_0$' such that
$$|z_n - z| < \varepsilon \quad \text{whenever } n > n_0$$

When the limit exists, the sequence is said to converge to $z$. We write
$$\lim_{n\to\infty} z_n = z.$$

If the sequence has no limit, it diverges.

**Theorem:** Suppose that $z_n = x_n + iy_n$ $(n = 1, 2, \ldots)$ and $z = x + iy$. Then
$$\lim_{n\to\infty} z_n = z \quad \text{iff} \quad \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y.$$

**Proof.** Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} x_n = x$, $\exists$ $n_1 \in \mathbb{N}$ such that
$$|x_n - x| < \varepsilon/2 \text{ whenever } n > n_1$$

Again, since $\lim_{n\to\infty} y_n = y$, $\exists$ $n_2 \in \mathbb{N}$ such that
$$|y_n - y| < \varepsilon/2 \text{ whenever } n > n_2$$

Let $n_0 = \max\{n_1, n_2\}$

Then $|x_n - x| < \varepsilon/2$ and $|y_n - y| < \varepsilon/2$ whenever $n > n_0$

Now,
$$|z_n - z| = |(x_n + iy_n) - (x + iy)|$$
$$= |(x_n - x) + i(y_n - y)|$$
$$\leq |x_n - x| + |y_n - y|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \text{whenever } n > n_0$$

Since $\varepsilon > 0$ is arbitrary

$\therefore \boxed{\lim_{n\to\infty} z_n = z.}$

Conversely,

Suppose $\lim_{n\to\infty} z_n = z$

Then for each $\varepsilon > 0$ $\exists\, n_0 \in \mathbb{N}$ such that

$|z_n - z| < \varepsilon$ whenever $n > n_0$

i.e. $|(x_n + iy_n) - (x + iy)| < \varepsilon$ whenever $n > n_0$ — (1)

But
$$|x_n - x| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$$

and
$$|y_n - y| \leq |(x_n - x) + i(y_n - y)| = |(x_n + iy_n) - (x + iy)|$$

$\therefore$ From (1), we get

$|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ whenever $n > n_0$

$\therefore \boxed{\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y}$

Note:

The theorem enables us to write

$$\boxed{\lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n}$$

Example 1. Show that $\lim_{n\to\infty} z_n = i$, where $z_n = \frac{1}{n^3} + i$, $n \in \mathbb{N}$

1st method (By defn.)

Let $\varepsilon > 0$ be given

Consider $|z_n - i| = \left|\frac{1}{n^3} + i - i\right| = \frac{1}{n^3}$

Choose $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\sqrt[3]{\varepsilon}}$

Then $|z_n - i| < \varepsilon$ whenever $n > n_0$.

$\therefore \lim_{n\to\infty} z_n = i$

2nd Method (By ~~Formula~~ Theorem).

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(\frac{1}{n^3} + i\right) = \lim_{n\to\infty}\frac{1}{n^3} + i.\lim_{n\to\infty} 1$$

$$= 0 + i.1 = i.$$

Example 2.

Let $z_n = -2 + i\,\frac{(-1)^n}{n^2}$ $(n = 1, 2, \ldots)$

By Theorem,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(-2 + i\,\frac{(-1)^n}{n^2}\right)$$

$$= \lim_{n\to\infty}(-2) + i\lim_{n\to\infty}\frac{(-1)^n}{n^2}$$

$$= -2 + i.0$$

$$= -2.$$

**Note:** If ~~we~~ we use polar coordinates,

Let us write $r_n = |z_n|$ and $\Theta_n = \text{Arg } z_n$, $(n = 1, 2, \ldots)$

where $\text{Arg } z_n$ denotes principal arguments $(-\pi < \Theta \leq \pi)$ of $z_n$.

Then, $\lim_{n\to\infty} r_n = \lim_{n\to\infty} \sqrt{4 + \frac{1}{n^4}} = 2$

But $\lim_{n\to\infty} \Theta_{2n} = \pi$

and $\lim_{n\to\infty} \Theta_{2n-1} = -\pi$

$(n = 1, 2, 3, \ldots)$

(Note that:
$z_{2n} = -2 + i\frac{1}{4n^2}$
$z_{2n-1} = -2 - i\frac{1}{(2n-1)^2}$)

We observe that lim of $\Theta_n$ doesn't exist as $n \to \infty$.

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Q1. Use defn of limit of sequences to verify that

$$\lim_{n\to\infty} z_n = -2 \text{ where } z_n = -2 + i\frac{(-1)^n}{n^2}$$

$(n = 1, 2, \ldots)$

Soln. Let $\varepsilon > 0$ be given

Consider, $|z_n - (-2)| = \left| i\frac{(-1)^n}{n^2} \right| = \frac{1}{n^2}$

By Archimedean property, we can find $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\sqrt{\varepsilon}}$

Then $|z_n - (-2)| < \varepsilon$ whenever $n > n_0$

Since $\varepsilon > 0$ is arbitrary, $\lim_{n\to\infty} z_n = -2$.

Q2. Let $\theta_n$ $(n=1,2,\dots)$ denote the principal arguments $(-\pi < \theta_n \le \pi)$ of the numbers

$$z_n = 2 + i\frac{(-1)^n}{n^2}, \quad (n=1,2,3,\dots)$$

Point out why $\lim_{n\to\infty} \theta_n = 0$.

Soln.

$$\theta_n = \tan^{-1}\left(\frac{(-1)^n}{2n^2}\right)$$

$$\lim_{n\to\infty} \theta_{2n} = 0$$

$$\lim_{n\to\infty} \theta_{2n-1} = 0$$

(Note that $z_{2n} = 2 + i\frac{1}{n^2}$, $z_{2n-1} = 2 - i\frac{1}{(2n-1)^2}$)

$$\therefore \lim_{n\to\infty} \theta_n = 0 \quad (n=1,2,\dots)$$

Q3. Show that, if $\lim_{n\to\infty} z_n = z$, then $\lim_{n\to\infty} |z_n| = |z|$.

Soln. Suppose that $\lim_{n\to\infty} z_n = z$.

Then for each $\varepsilon > 0$ $\exists$ a +ve integer $n_0$ such that

$$|z_n - z| < \varepsilon \quad \text{whenever } n > n_0 \qquad \text{--- (1)}$$

We know that

$$\big||z_n| - |z|\big| \le |z_n - z|$$

$$\therefore \big||z_n| - |z|\big| < \varepsilon \quad \text{whenever } n > n_0 \quad (\text{Using (1)})$$

Since $\varepsilon > 0$ is arb.

$$\lim_{n\to\infty} |z_n| = |z|.$$

~~Q6~~. Q5.
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Show that a limit of convergent sequence of Complex numbers is unique.

Do yourself (Easy).

Q9
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Every convergent sequence is bounded.

(a) Suppose $\lim_{n\to\infty} z_n = z$

For $\varepsilon = 1$, $\exists$ a +ve integer $n_0$ s.t.

$|z_n - z| < 1$ whenever $n > n_0$

Now, $|z_n| = |z + (z_n - z)|$

$\leq |z| + |z_n - z|$

$< |z| + 1$ whenever $n > n_0$

Let $M = \max\{|z_1|, |z_2|, \ldots |z_{n_0}|, |z|+1\}$

Then $|z_n| \leq M \quad \forall n \in \mathbb{N}$

(b) ~~Alte~~ Alternative Method

Suppose $\lim_{n\to\infty} z_n = z$.

Let $z_n = x_n + iy_n$ and $z = x + iy$

Then $\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Since $\{x_n\}$ and $\{y_n\}$ are convergent sequences of real numbers

$\therefore \exists M_1, M_2 > 0$ s.t.

$|x_n| \leq M_1, \; |y_n| < M_2 \quad \forall n \in \mathbb{N}$ (Every convergent sequences is bounded in theory of sequences of real no's.)

$\Rightarrow |x_n + iy_n| \leq |x_n| + |y_n| \leq M_1 + M_2 = M$ (say)

$\Rightarrow \boxed{|z_n| \leq M \quad \forall n \in \mathbb{N}}$

## Sec 56. Convergence of Series

An infinite series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$$

of complex numbers converges to the sum $S$ if the sequence

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N \quad (N = 1, 2, 3, \ldots)$$

of partial sums converges to $S$.

We then write $\displaystyle\sum_{n=1}^{\infty} z_n = S$.

We note that since a sequence can have atmost one limit, a series can have atmost one sum.

When a series doesn't converge, we say it diverges.

**Theorem:** Suppose that $z_n = x_n + i y_n$ $(n = 1, 2, 3, \ldots)$ and $S = X + iY$.

Then $\displaystyle\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n = X$ and $\displaystyle\sum_{n=1}^{\infty} y_n = Y$.

**Proof.** Let $S_N = X_N + i Y_N$ be sequence of partial sums of $\displaystyle\sum_{n=1}^{\infty} z_n$ where $X_N = \displaystyle\sum_{n=1}^{N} x_n$ & $Y_N = \displaystyle\sum_{n=1}^{N} y_n$

We know that

$$\lim_{N\to\infty} S_N = S \iff \lim_{N\to\infty} X_N = X \text{ and } \lim_{N\to\infty} Y_N = Y.$$

i.e. $\sum_{n=1}^{\infty} z_n = S \iff \sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

<u>Remark</u>: The theorem tells us that one can write

$$\sum_{n=1}^{\infty}(x_n + i y_n) = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

whenever it is known that the two series on the right converge or that the one on the left does.

<u>Corollary 1.</u> <u>If a series of complex numbers converges, the $n^{th}$ term converges to zero as $n$ tends to infinity.</u>

<u>Proof.</u> Assume that $\sum_{n=1}^{\infty} z_n$ converges

where $z_n = x_n + i y_n$

Then by ~~above~~ preceding theorem,

each of the series $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converges.

From the calculus of real numbers, we have

$x_n \to 0$ and $y_n \to 0$ as $n \to \infty$

$$\therefore \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0 + i \cdot 0 = 0.$$

<u>Corollary 2</u>: The absolute convergence of a series of complex numbers implies the convergence of that series.

<u>Proof.</u> Suppose that $\sum_{n=1}^{\infty} z_n$ converges absolutely

i.e. $\sum_{n=1}^{\infty} |z_n|$ converges.

$\Rightarrow \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$ Converges $\quad (z_n = x_n + iy_n)$

Since $|x_n| \leq \sqrt{x_n^2+y_n^2}$ and $|y_n| \leq \sqrt{x_n^2+y_n^2}$

$\therefore$ By Comparison Test

~~Series~~ $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ are convergent

$\Rightarrow \sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are cgt. (By Absolute convergence in Real Calculus)

$\Rightarrow \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$ Converges

$\Rightarrow \sum_{n=1}^{\infty} z_n$ converges.

<u>Remark</u>: In establishing the ~~fact~~ fact that the sum of a series is a given number $S$, it is often convenient to define remainder $\rho_N$ after $N$ terms, using the partial sums $S_N$

i.e. $\rho_N = S - S_N$

Thus $S = S_N + \rho_N$

and since $|S_N - S| = |\rho_N - 0|$.

we can see that

"a series converges to a number $S$ iff the sequence of remainders tends to zero".

**Example.** Show that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ whenever $|z|<1$

Let $S_N = \sum_{n=0}^{N-1} z^n = 1+z+z^2+ \cdots + z^{N-1} \quad (z \neq 1)$

be the sequence of partial sums of the series $\sum_{n=1}^{\infty} z_n$

Now, $S_N = \frac{1-z^N}{1-z}$

If $S = \frac{1}{1-z}$

Then $\rho_N(z) = S(z) - S_N(z) = \frac{1}{1-z} - \frac{1-z^N}{1-z}$

$$= \frac{z^N}{1-z} \qquad (z \neq 1)$$

$\therefore \ |\rho_N(z)| = \frac{|z|^N}{|1-z|}$

Since $|z|<1$

$\therefore \ |z|^N \to 0$ as $N \to \infty$

$\therefore \ \lim_{N\to\infty} |\rho_N(z)| \to 0$ as $N \to \infty$

$\therefore \ \boxed{\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \text{ whenever } |z|<1.}$

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Q4. Consider $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ whenever $|z|<1$ — (1)

Let $z = re^{i\theta}$ where $0<r<1$

Then show that

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

when $0<r<1$.

Soln. Since $z = re^{i\theta}$ where $0<r<1$

$$\therefore |z| = r < 1$$

Then $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ (from (1)) — (*)

(*) Can further be written as

$$1 + \sum_{n=1}^{\infty} z^n = \frac{1}{1-z}$$

$$\Rightarrow \sum_{n=1}^{\infty} z^n = \frac{1}{1-z} - 1 = \frac{z}{1-z} \quad \text{— (**)}$$

Consider L.H.S. of (**)

$$\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty} (re^{i\theta})^n = \sum_{n=1}^{\infty} r^n e^{in\theta}$$

$$= \sum_{n=1}^{\infty} r^n (\cos n\theta + i\sin n\theta)$$

$$= \sum_{n=1}^{\infty} r^n \cos n\theta + i\sum_{n=1}^{\infty} r^n \sin n\theta$$

Now, Consider R.H.S. of (**),

$$\frac{z}{1-z} = \frac{re^{i\theta}}{1-re^{i\theta}} = \frac{re^{i\theta}}{1-re^{i\theta}} \times \frac{1-re^{-i\theta}}{1-re^{-i\theta}}$$

$$= \frac{re^{i\theta} - r^2}{1 - r(e^{i\theta} + e^{-i\theta}) + r^2}$$

$$= \frac{r\cos\theta - r^2 + ir\sin\theta}{1 - 2r\cos\theta + r^2}$$

Thus,

$$\sum_{n=1}^{\infty} r^n \cos n\theta + i \sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} + i \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

Equating real and imaginary parts, we get

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2}$$

and $$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}.$$

Q6. Show that if $\sum_{n=1}^{\infty} z_n = S$ then $\sum_{n=1}^{\infty} \bar{z}_n = \bar{S}$.

Soln. Suppose that $\sum_{n=1}^{\infty} z_n = S$

Let $z_n = x_n + iy_n$

and $S = X + iY$

Then by Theorem $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$.

Consider

$$\sum_{n=1}^{\infty} \bar{z}_n = \sum_{n=1}^{\infty} (x_n - iy_n) = \sum_{n=1}^{\infty} [x_n + i(-y_n)] = X - iY = \bar{S}$$

$$\left(\because \sum_{n=1}^{\infty} (-y_n) = -\sum_{n=1}^{\infty} y_n = -Y\right)$$

**Q7.** Let 'c' denote any complex number. Show that if $\sum_{n=1}^{\infty} z_n = S$ then $\sum_{n=1}^{\infty} c z_n = cS$

Try yourself.

**Q8.** Show that if $\sum_{n=1}^{\infty} z_n = S$ and $\sum_{n=1}^{\infty} w_n = T$ then $\sum_{n=1}^{\infty} (z_n + w_n) = S+T.$

**Soln.** Suppose that $\sum_{n=1}^{\infty} z_n = S$ and $\sum_{n=1}^{\infty} w_n = T$

Let $z_n = x_n + iy_n$, $S = X + iY$

and $w_n = u_n + iv_n$, $T = U + iV$

Since $\sum_{n=1}^{\infty} z_n = S$, we have

$$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y$$

Again, since $\sum_{n=1}^{\infty} w_n = T$, we have

$$\sum_{n=1}^{\infty} u_n = U \text{ and } \sum_{n=1}^{\infty} v_n = V$$

Since $\sum_{n=1}^{\infty} (x_n + u_n) = X + U$

and $\sum_{n=1}^{\infty} (y_n + v_n) = Y + V$

$$\therefore \sum_{n=1}^{\infty} \left[ (x_n + u_n) + i (y_n + v_n) \right] = (X+U) + i(Y+V)$$

i.e. $\sum_{n=1}^{\infty} \left[ (x_n + iy_n) + (u_n + iv_n) \right] = (X+iY) + (U+iV)$

or $\sum_{n=1}^{\infty} (z_n + w_n) = S + T.$