02.02APS

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Chapter - 3 (Sequences and Series)

Sequence:

A sequence of real numbers (or real sequence) is a function from set of natural numbers ($\mathbb{N}$) to set of real numbers ($\mathbb{R}$).

or

A sequence is a f^n from $\mathbb{N}$ to $\mathbb{R}$.

If $X : \mathbb{N} \to \mathbb{R}$ is a sequence, then $X(n) \in \mathbb{R}$ $\forall$ $n \in \mathbb{N}$.

Notation:

If $x : \mathbb{N} \to \mathbb{R}$ is a sequence.

$\{x_n : n \in \mathbb{N}\}$ or $\{x_n\}$ or (x_n) or $\langle x_n \rangle$.

or $(x_n : n \in \mathbb{N})$

or $(x_n : n \in \mathbb{N})$

$\boxed{(x_n) \text{ or } \langle x_n \rangle \text{ or } \{x_n\} \Rightarrow \text{Mainly we shall take.}}$

Example: ① $\left(\frac{1}{n}\right)$ is a sequence.

$$\left(\frac{1}{n}\right) = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\}.$$

② $\{(-1)^n\}$ is a sequence.

$$\{(-1)^n\} = \{-1, 1, -1, 1, -1, 1, \cdots\}$$

③ The Fibonacci Sequence:

It is defined as $F = \{f_n\}$,

where $f_1 = 1$, $f_2 = 1$ and $f_{n+1} = f_{n-1} + f_n \ \forall n \geq 2$

$$F = \{f_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \cdots\}.$$

Number of leaves in a tree

Number of leaves in a tree generally follows fibonacci sequence.

Limit of a sequence:

A sequence $X = (x_n)$ is said to converge to a point $x \in \mathbb{R}$, if for every $\epsilon > 0$, there exists a natural number $K(\epsilon)$ such that $|x_n - x| < \epsilon \quad \forall\, n \geq K$.

Notation: $\boxed{\lim (x_n) = x.}$

Example: $\lim \left(\frac{1}{n}\right) = 0.$

$x_n = \frac{1}{n}$

Let $\epsilon > 0$ and $|x_n - 0| < \epsilon$

$\Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon$

$\Rightarrow \frac{1}{n} < \epsilon$

$\Rightarrow n > \frac{1}{\epsilon} = K.$

Therefore, we have for all $n > K(\epsilon)$

$$\left|\frac{1}{n} - 0\right| < \epsilon$$

$$\Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon \quad \forall n > K$$

$$\Rightarrow \lim\left(\frac{1}{n}\right) = 0.$$

Note:

(i) If a sequence (x_n) has a limit, we say that (x_n) is a convergent sequence.

(ii) If a sequence (x_n) has no limit, we say that (x_n) is a divergent sequence.

Theorem: (Uniqueness of limit):

A sequence in $\mathbb{R}$ can have atmost one limit

or

Limit of a sequence is unique, if it exists.

Proof: Suppose that a sequence (x_n) has two limits x' and x''.

Now, $\lim (x_n) = x' \Rightarrow \forall \frac{\epsilon}{2} > 0$, $\exists K'$ such that $|x_n - x'| < \epsilon/2 \quad \forall n \geq K'$ —— ①

If $\epsilon > 0$
$\Rightarrow \epsilon/2 > 0$.

Now, $\lim (x_n) = x''$
$\Rightarrow \forall \epsilon/2 > 0$, $\exists K''$ such that
$|x_n - x''| < \epsilon/2 \quad \forall n \geq K''$ —— ②

Now, let $K = \max\{K', K''\}$
from eqn ① & ②, we have.
$|x_n - x'| < \epsilon/2 \quad \forall n \geq K$ —— ③
and $|x_n - x''| < \epsilon/2 \quad \forall n \geq K$ —— ④

$x \geq 5$, $x \geq 10$
$z = \max\{5, 10\} = 10$
$x \geq 10$. $x \geq z$

Now, $|x' - x''| = |x' - x_n + x_n - x''|$

$\leq |x' - x_n| + |x_n - x''|$

$= |x_n - x'| + |x_n - x''|$

$< \epsilon/2 + \epsilon/2$ [using eq (3) & (4)
$\forall n \geq k$

$= \epsilon \quad \forall n \geq k$.

$\therefore |x' - x''| < \epsilon \quad \forall n \geq k$

$\Rightarrow |x' - x''| < \epsilon$, where $\epsilon > 0$.

$\Rightarrow |x' - x''| = 0$

$\Rightarrow x' - x'' = 0$

$\Rightarrow x' = x''$

$\therefore$ limit of sequence (x_n) is unique.