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Superposition of Harmonic Oscillations

Superposition Principle :-

It states that the resultant of two or more harmonic displacements is simply the algebraic sum of the individual displacements.

Linearity :-

For small oscillations, the restoring force F is proportional to displacement ψ ($F = -k\psi$) and the resulting oscillations are harmonic.

eq^n of motion will be :- $\frac{d^2\psi}{dt^2} = -k\psi$

Here $\frac{d^2\psi}{dt^2} \propto \psi$ → It doesn't depend on higher power of ψ

$\therefore$ eq^n is linear.

⇒ A linear differential eq^n is the one that contain terms that depend only on the first powers of the variable and derivatives.

→ But, if the oscillations are very violent i.e. with a large amplitude.

e.g. In a simple pendulum, for this case $\sin\alpha \neq \alpha$.

∴ Restoring force will depend on ψ, ψ^2, ψ^3 etc.

⇒ Restoring force will not be linear in ψ and eq^n of motion will be non-linear.

Note :- Non-linear eq's are difficult to solve.

→ An eq^n is said to be homogenous if it contain no terms independent of the variable ψ.

Property of Linear homogenous diff eq^n

The sum of any two solutions is itself a solution. this is nothing but the superposition principle.

Superpos^n principle apply only to linear differential eq^{n}s. It doesn't apply if the eq's are non-linear.

Let us say that the oscillations of a system with one degree of freedom:-

$$\frac{d^2\psi}{dt^2} = -\omega^2\psi + \alpha\psi^2 + \beta\psi^3 + \text{---} \quad (1)$$

→ This is eq^n of simple pendulum in the cond^n $\sin\alpha \neq \alpha$.

eq^n (1) is non-linear ∵ it contains terms like ψ^2, ψ^3 etc.

When α & β are zero then eq^n (1) can be linear and homogenous

→ Although it is difficult to solve eq^n (1) But we try to do so.

Let us assume $\psi_1(t)$ and $\psi_2(t)$ are two sol^{n}s of eq^n.

$$\Rightarrow \frac{d^2\psi_1}{dt^2} = -\omega^2\psi_1 + \alpha\psi_1^2 + \beta\psi_1^3 + \text{---} \quad (2)$$

$$\frac{d^2\psi_2}{dt^2} = -\omega^2\psi_2 + \alpha\psi_2^2 + \beta\psi_2^3 + \text{---} \quad (3)$$

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Let us assume $\psi = \psi_1 + \psi_2$ be ~~the~~ also the sol^n of eq^n (1) by superpos^n principle.

$$\Rightarrow \frac{d^2(\psi_1+\psi_2)}{dt^2} = -\omega^2(\psi_1+\psi_2) + \alpha(\psi_1+\psi_2)^2 + \beta(\psi_1+\psi_2)^3 + \cdots \quad \text{— (4)}$$

Adding eq^{n}s (2) & (3) $\Rightarrow$

$$\frac{d^2\psi_1}{dt^2} + \frac{d^2\psi_2}{dt^2} = -\omega^2\psi_1 - \omega^2\psi_2 + \alpha\psi_1^2 + \alpha\psi_2^2 + \beta\psi_1^3 + \beta\psi_2^3 + \cdots \quad \text{— (5)}$$

If our assumpt^n is correct then eq^n (4) & (5) should be same.

But these eq^n agree if $\rightarrow$

$$\frac{d^2}{dt^2}(\psi_1+\psi_2) = \frac{d^2\psi_1}{dt^2} + \frac{d^2\psi_2}{dt^2}$$

$$-\omega^2(\psi_1+\psi_2) = -\omega^2\psi_1 - \omega^2\psi_2$$

$$\alpha(\psi_1+\psi_2)^2 \neq \alpha(\psi_1^2+\psi_2^2)$$

$$\beta(\psi_1+\psi_2)^3 \neq \beta(\psi_1^3+\psi_2^3)$$

Here first two conditions are true, but the last two are not true unless α and β are zero.

$\Rightarrow$ Our assumpt^n $\psi = \psi_1 + \psi_2$ is a solution can be realized if ~~and~~ and only if the non-linear terms α and β are absent.

$\Rightarrow$ Superp^n of two solutions is itself a solution if and only if the eq^n is linear.

Importance of Superposition Principle

Consider a simple pendulum with small oscillation. It's eq^n of motion will be :-

$$\frac{d^2\psi}{dt^2} = -\omega^2 \psi \rightarrow \text{linear}$$

~~Initial Cond^{n}s are~~

Under Initial Cond^{n}s, let the sol^n of the eq be ψ_1.

$$\Rightarrow \psi_1 = A_1 \cos(\omega t + \phi_1)$$

A_1 & ϕ_1 are determined from initial cond^{n}s

$\rightarrow$ angular freq (ω) doesn't depend on initial condⁿ.

$\rightarrow$ Under another set of initial cond'ⁿs ; the displacement ψ_2 :-

$$\psi_2 = A_2 \cos(\omega t + \phi_2)$$

A_2, ϕ_2 are determined from new set of initial cond'ⁿs.

$\rightarrow$ Now we superpose the initial conditions corresponding to ψ_1 & ψ_2. In other words, we given an initial displacement = initial disp(ψ_1) + " disp(ψ_2)

$\rightarrow$ Give an initial vel = initial vel (ψ_1) + " " (ψ_2).

Now from superposⁿ principle, the new motion described by ψ_3 will be superposⁿ of ψ_1 and ψ_2

$$\Rightarrow \psi_3 = \psi_1 + \psi_2.$$

So, there is no need to solve the eqⁿ of motion to find the new motion.

$\Rightarrow$ Resultant of two or more harmonic displacements is given by the algebraic sum of the individual displacements.

Note:- This result holds only if eqⁿ of motion is linear.

Following are important cases of the superposⁿ of harmonic oscillations :—

(1) Two oscillations of the same frequency moving in the same direction (interference of oscillations)

(2) Two oscillations of slightly different freq. moving in the same dirⁿ. (beats)

(3) Two oscillations of the same freq moving in opposite dirⁿ (Standing waves)

(4) Two oscillations of same freq but perpendicular to each other (Lissajous fig)