

(20)

Every join homo. is an order homomorphism.

Pf let $f: L \rightarrow M$ be a join homo.
let $a \leq b$ in L

$$\Rightarrow a \vee b = b$$

$$\Rightarrow f(a \vee b) = f(b)$$

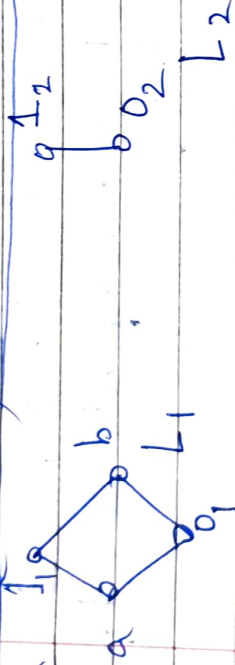
$$\Rightarrow f(a) \vee f(b) = f(b)$$

$$\Rightarrow f(a) \leq f(b)$$

$\therefore f$ is an order homo.

Every meet homo. is an order homomorphism.

(Prove)



$$f: L_1 \rightarrow L_2$$

$$f(0_1) = f(c) = f(b) = 0_2$$

$$f(1_1) = 1_2$$

Verify: f is an order homo.

Verify: f is a meet homo.

$$f(xy) = f(x) \wedge f(y) \quad \forall x, y \in L_1$$

$$\text{Verify: } f(a \vee b) \neq f(a) \vee f(b)$$

f is not a join homo.

Defn 2 Posets P & Q are order

isomorphic ($P \cong Q$) if $\exists$ a map

$f: P \rightarrow Q$ s.t. f is onto

& $x \leq y$ in P iff $f(x) \leq f(y)$ $\forall x, y \in P$

(order preserving both ways)

If P & Q are order isomorphic posets then ϕ is 1-1

If let $\phi(x) = \phi(y)$

By reflexivity, $\phi(x) \leq \phi(y)$ & $\phi(y) \leq \phi(x)$ in Q

$\Rightarrow x \leq y$ & $y \leq x$ in P ($\because \phi$ is order isomorphism)

$\Rightarrow x = y$

$\Rightarrow \phi$ is 1-1

ex $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ w.r.t ' $\leq$ ' is a poset (chain)

Define $\phi: \mathbb{N}_0 \rightarrow \mathbb{N}$ as

$$\phi(x) = x + 1$$

Then ϕ is 1-1, onto, order preserving both ways.

$$\therefore (\mathbb{N}_0, \leq) \cong (\mathbb{N}, \leq)$$

Thm: Lattices isomorphic as algebras are also isomorphic as posets.

If let L, M be iso. as algebras,
 $\Rightarrow \exists \phi: L \rightarrow M$ which is 1-1, onto & lattice homomorphism.

Let $x \leq y$ in L

$$\Rightarrow x \wedge y = x$$

$$\Rightarrow \phi(x \wedge y) = \phi(x) \wedge \phi(y)$$

$$\Rightarrow \phi(x) \wedge \phi(y) = \phi(x) \text{ in } M$$

$$\Rightarrow \phi(x) \leq \phi(y) \text{ in } M.$$

Civslly let $x' \leq y'$ in M

Since ϕ is bijective, $\exists$ Unique elts. $x, y \in L$ s.t. $\phi(x) = x', \phi(y) = y'$.

$$x' \leq y' \text{ in } M \Rightarrow \phi(x) \leq \phi(y) \text{ in } M$$

$$\Rightarrow \phi(x) \wedge \phi(y) = \phi(x) \text{ in } M$$

$$\Rightarrow \phi(xy) = \phi(x)$$

$$\Rightarrow xy = x \text{ in } L$$

$$\Rightarrow x \leq y$$

$\therefore \phi$ preserves order both ways

$$\Rightarrow L \cong M$$

Thm Lattices iso. as posets are also iso. as algebras.

If let $\phi: L \rightarrow M$ be an order iso.

Consider $xy \leq x$ in L
& $xy \leq y$ in L

$$\Rightarrow \phi(xy) \leq \phi(x)$$

$$\& \phi(xy) = \phi(y)$$

$\Rightarrow \phi(xy)$ is a l.b. of $\phi(x)$ & $\phi(y)$ in M

Let h be any l.b. of $\phi(x)$ & $\phi(y)$ in M

$$h \leq \phi(x), h \leq \phi(y)$$

Since ϕ is bijective

$\therefore \exists$ unique $g \in L$ s.t. $\phi(g) = h$.

$$\Rightarrow \phi(g) \leq \phi(x) \& \phi(g) \leq \phi(y)$$

Since ϕ is order iso,

$$\therefore g \leq x, g \leq y \text{ in } L$$

$$\Rightarrow g \text{ is a l.b. of } x \& y \text{ in } L$$

$$\Rightarrow g \leq \text{glb of } x \& y = xy$$

$$\Rightarrow \phi(g) \leq \phi(xy)$$

$$\Rightarrow h \leq \phi(xy)$$

$$\therefore \phi(xy) = \text{glb of } \phi(x) \& \phi(y)$$

$$= \phi(x) \wedge \phi(y)$$

Similarly, prove $\phi(xy) = \phi(x) \vee \phi(y)$

Defn Cross product (Cartesian Product)
 Let L, M be lattices.

$$L \times M = \{ (a, b) \mid a \in L, b \in M \}$$

Define $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ in L
 & $y_1 \leq y_2$ in M .

Prove $\leq$ is a Partial order relation.
 an $L \times M$

$\therefore (L \times M, \leq)$ is a poset.

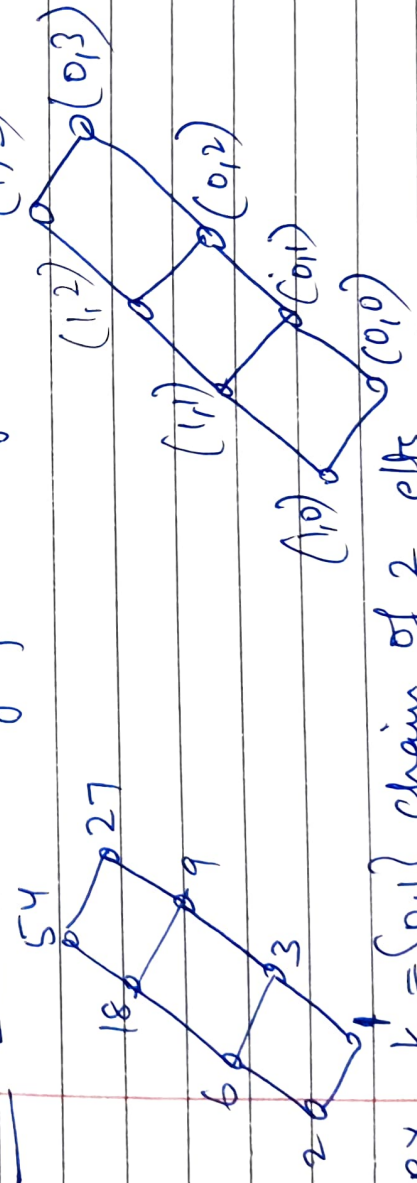
Define: $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$
 & $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$

Clearly $\wedge$ & $\vee$ are binary Comps.
 in $L \times M$.

Prove $\wedge$ & $\vee$ are Commutative
 associative & Absorption laws hold.

$\therefore (L \times M, \leq, \wedge, \vee)$ is a lattice.

ex $L =$ lattice of factors of 54 $(1, 3)$



ex $K_1 = \{0, 1\}$ chain of 2 elts
 $K_2 = \{0, 1, 2\}$ chain of 3 elts.
 $K_1 \times K_2$ is a lattice.

$\phi: L \rightarrow K_1 \times K_2$
 $\phi(2^{\alpha} 3^{\beta}) = (\alpha, \beta) \quad \forall 2^{\alpha} 3^{\beta} \in L$
 ϕ is 1-1, onto, order pres. both ways.
 $\therefore L \cong K_1 \times K_2$