$$v = \sqrt{\frac{2\pi T}{\lambda\rho}}$$

These waves are called *ripples* and propagate mainly due to surface tension. If λ lies between these two values both terms are included to determine the velocity. Both for very large and very small values of λ, the velocity of the phase wave tends to infinity. Thus there must be wavelength of intermediate value, i.e. critical wavelength for which velocity of water waves is minimum.

2.6 PLANE WAVES

The palne wave is the simplest example of a three dimensional wave. It exists at a given time, when all the surfaces on which a disturbance has constant phase, forms a set of planes, each generally perpendicular to the direction of propagation. That is, *a plane wave is defined as a wave in which the wave amplitude is constant over all points of a plane perpendicular to the direction of propagation.*

There are quite practical reasons for studying this sort of disturbance, one of which is that by using optical devices, we can readily produce light resembling planc waves.

The mathematical expression for a plane that is perpendicular to a given vector **k** and that passes through some point (x_0, y_0, z_0) is rather easy to derive (Fig. 2.15). First we write the position vector in Cartesian coordinates in terms of the unit basis vectors (Fig. 2.15a)

$$\mathbf{r} = x\boldsymbol{i} + y\boldsymbol{i} + z\boldsymbol{k}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are unit vector along respective axes.

It begins at some arbitrary origin O and ends at the point (x, y, z), which can, for the moment, be anywhere in space.

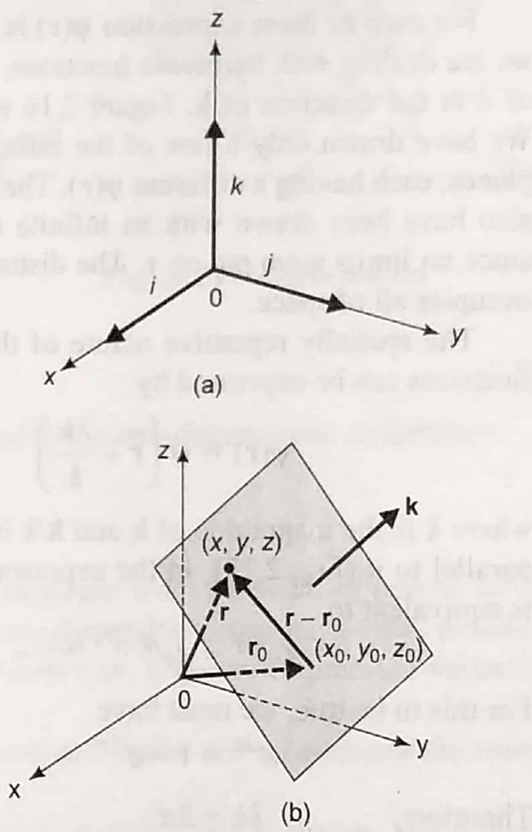

Fig. 2.15 *(a) The Cartesian unit basis vectors. (b) A plane wave moving in the* **k** *direction.*

Similarly,

$$(\mathbf{r} - \mathbf{r}_0) = (x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{j} + (z - z_0)\boldsymbol{k}$$

By setting $$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{k} = 0 \quad \ldots(2.58)$$

we force the vector $(\mathbf{r} - \mathbf{r}_0)$ to sweep out a plane perpendicular to $\mathbf{k}$, as its endpoint (x, y, z) takes on all allowed values. With

$$\mathbf{k} = k_x\boldsymbol{i} + k_y\boldsymbol{j} + k_z\boldsymbol{k} \quad \ldots(2.59)$$

Equation (2.58) can be expressed in the form

$$k_x(x - x_0) + k_y(y - y_0) + k_z(z - z_0) = 0 \quad \ldots(2.60)$$

or as $$k_x x + k_y y + k_z z = a \quad \ldots(2.61)$$

where

$$a = k_x x_0 + k_y y_0 + k_z z_0 = \text{constant} \qquad \ldots(2.62)$$

The most concise form of the equation of a plane perpendicular to **k** is then just

$$\mathbf{k} \cdot \mathbf{r} = \text{constant} = a \qquad \ldots(2.63)$$

The plane is the locus of all points whose position vectors have the same projection onto the **k** direction.

We can now construct a set of planes over which $\psi(\mathbf{r})$ varies in space sinusoidally, namely,

$$\psi(\mathbf{r}) = A \sin(\mathbf{k} \cdot \mathbf{r}) \qquad \ldots(2.64)$$

$$\psi(\mathbf{r}) = A \cos(\mathbf{k} \cdot \mathbf{r}) \qquad \ldots(2.65)$$

or

$$\psi(\mathbf{r}) = A e^{i\mathbf{k} \cdot \mathbf{r}} \qquad \ldots(2.66)$$

For each of these expression $\psi(\mathbf{r})$ is constant over every plane defined by $\mathbf{k} \cdot \mathbf{r}$ = constant. Since we are dealing with harmonic functions, they should repeat themselves in space after a displacement of λ in the direction of **k**. Figure 2.16 is a rather humble representation of this kind of expression. We have drawn only a few of the infinite number of planes, each having a different $\psi(\mathbf{r})$. The planes should also have been drawn with an infinite spatial extent, since no limits were put on **r**. The disturbance clearly occupies all of space.

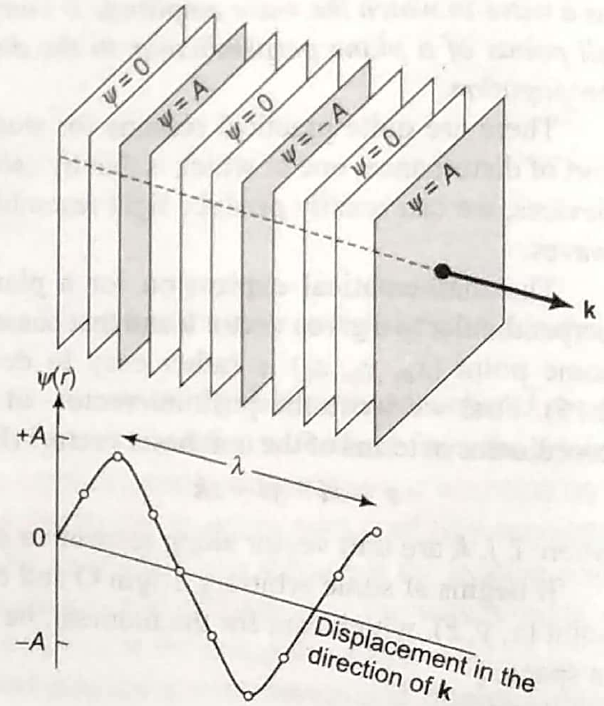

Fig. 2.16 *Wavefronts for a harmonic plane wave*

The spatially repetitive nature of these harmonic functions can be expressed by

$$\psi(\mathbf{r}) = \psi\left(\mathbf{r} + \frac{\lambda \mathbf{k}}{k}\right) \qquad \ldots(2.67)$$

where k is the magnitude of **k** and $\mathbf{k}/k$ is a unit vector parallel to it (Fig. 2.17). In the exponential form, this is equivalent to

$$Ae^{\mathbf{k} \cdot \mathbf{r}} = Ae^{i\mathbf{k} \cdot (\mathbf{r} + \lambda \mathbf{k}/k)} = Ae^{i\mathbf{k} \cdot \mathbf{r}} e^{i\lambda k}$$

For this to be true, we must have

$$e^{i\lambda k} = 1 = e^{i2\pi}$$

Therefore, $\quad \lambda k = 2\pi$

and $\quad k = 2\pi/\lambda$

The vector **k**, whose magnitude is the *propagation number k*, is called the *propagation vector*.

At any fixed point in space where **r** is constant, the phase is constant as is $\psi(\mathbf{r})$; in short, the planes are motionless. To get things moving, $\psi(\mathbf{r})$ must be made to vary in time, something we can accomplish by introducing the time dependence is an analogous fashion to that of the one-dimensional wave. Here then

$$\psi(\mathbf{r}, t) = Ae^{i(\mathbf{k} \cdot \mathbf{r} \mp \omega t)} \qquad \ldots(2.68)$$

with A, ω and k as constant. As this disturbance travels along in the **k**-direction, we can assign a phase corresponding to it at each point in space and time. At any given time, *the surfaces joining all points of equal phase are known as wavefronts*. Note that wavefunction will have a constant value over the wavefront only if the amplitude A has a fixed value at every point on the wavefront. In general, A is a function of **r** and may not be constant over all space or even over a wavefront. In the latter case, the wave is said to be *inhomogeneous*. We will not be concerned with this sort of disturbance.

The phase velocity of a plane wave given by eqn. (2.68) is equivalent to the propagation velocity of the wavefront. In Fig. 2.17, the scalar component of **r** in the direction of **k** is $\mathbf{r}_k$. The disturbance on a wavefront is constant, so that after time dt, if the front moves along **k** a distance dr_k, we must have

$$\psi(\mathbf{r}, t) = \psi(r_k + dr_k, t + dt) = \psi(r_k, t) \quad \ldots(2.69)$$

In exponential form, this is

$$Ae^{i(\mathbf{k}\cdot\mathbf{r} \pm \omega t)} = Ae^{i(kr_k + kdr_k \mp \omega t \mp \omega dt)} = Ae^{i(kr_k \mp \omega t)}$$

and so it must be that $kdr_k = \mp \omega dt$

The magnitude of the wave velocity, dr_k/dt, is then

$$\frac{dr_k}{dt} = \pm \frac{\omega}{k} = \pm v \quad \ldots(2.70)$$

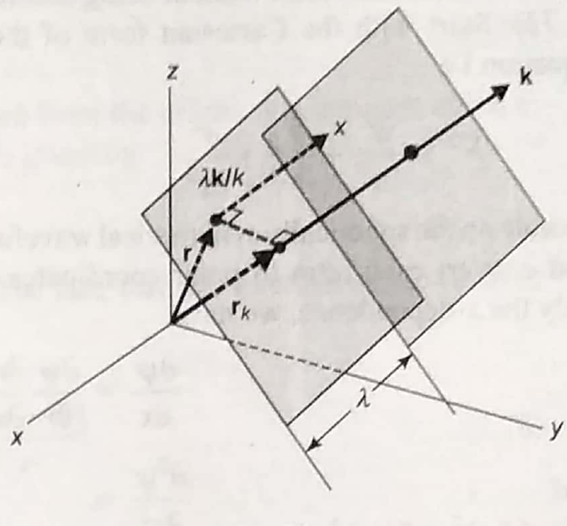

Fig. 2.17 *Plane waves*

We could have anticipated this result by rotating the coordinate system in Fig. 2.17 so that **k** was parallel to the x-axis. For that orientation

$$\psi(r, t) = Ae^{i(kx \mp \omega t)} \quad \ldots(2.71)$$

since $\mathbf{k} \cdot \mathbf{r} = kr_k = kx$. The wave has been effectively reduced to the one-dimensional disturbance.

2.7 SPHERICAL WAVES

Toss a stone into a tank of water. The surface ripples that originate from the point of impact spread out in two dimensional circular waves. Extending this to three dimensions, imagine a small pulsating spheres surrounded by a fluid. As the source expands and contracts, it generates pressure variations that propagate outward as spherical waves.

"*Spherical waves are waves in which the surfaces of common phase are spheres and the source of waves is a central point.*"

Consider an ideal point source of light. The radiation originating from it streams out radially, uniformly in all directions. The source is said to be isotropic and the resulting wavefronts are again concentric spheres that increase in diameter as they expand out into the surrounding space. The obvious symmetry of the wavefronts suggests that is might be more convenient to describe them in terms of spherical polar coordinates, Fig. 2.18. In this representation the Laplacian operator is

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial\phi^2} \quad \ldots(2.72)$$

where r, θ, ϕ are defined by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

Remember that we are looking for a description of spherical waves, waves that are spherically symmetrical (i.e., ones that do not depend on θ and ϕ) so that

$$\psi(\mathbf{r}) = \psi(r, \theta, \phi) = \psi(r)$$

The Laplacian of $\psi(r)$ is then simply

$$\nabla^2\psi(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \qquad \ldots(2.73)$$

We can obtain this result without being familiar with eqn. (2.72). Start with the Cartesian form of the Laplacian equation i.e

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

operate on the spherically symmetrical wavefunciton $\psi(r)$, and convert each term to polar coordinates. Examining only the x-dependence, we have

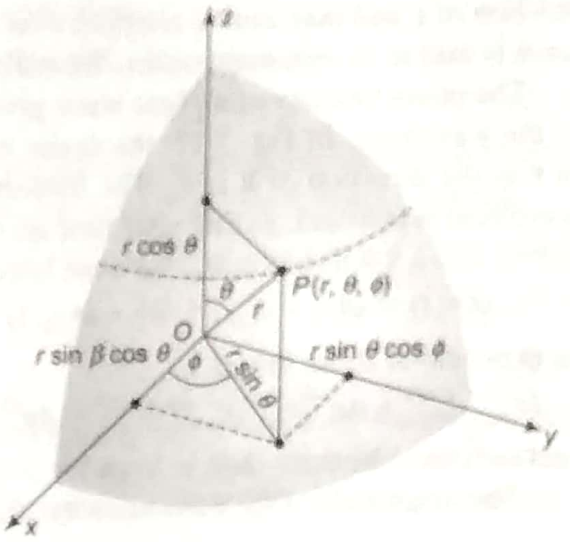

Fig. 2.18 *The geometry of spherical coordinates.*

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x}$$

and

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial\psi}{\partial r}\frac{\partial^2 r}{\partial x^2}$$

since $\qquad \psi(\mathbf{r}) = \psi(r)$

Using $\qquad x^2 + y^2 + z^2 = r^2$

we have

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{1}{r}\frac{\partial}{\partial x}x + x\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{1}{r}\left(1 - \frac{x^2}{r^2}\right)$$

and so

$$\frac{\partial^2\psi}{\partial x^2} = \frac{x^2}{r^2}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\left(1 - \frac{x^2}{r^2}\right)\frac{\partial\psi}{\partial r}$$

Now having $\partial^2\psi/\partial x^2$, we form $\partial^2\psi/\partial y^2$ and $\partial^2\psi/\partial z^2$, and on adding get

$$\nabla^2\psi(r) = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r}$$

which is equivalent to eqn. (2.73). This result can be expressed in a slightly different form:

$$\nabla^2\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) \qquad \ldots(2.74)$$

The differential wave equation can then be written as

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \qquad \ldots(2.75)$$

Multiplying both sides by r, yields

$$\frac{\partial^2}{\partial r^2}(r\psi) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(r\psi) \qquad \ldots(2.76)$$

Notice that this expression is now just the one-dimensional differential wave equation, where the space variable is r and the wavefunction is the product $(r\psi)$. The solution of eqn. (2.76) is then simply

$$r\psi(r, t) = f(r - vt)$$

or
$$\psi(r, t) = \frac{f(r - vt)}{r} \quad ...(2.77)$$

This represents a spherical wave progressing radially outward from the origin, at a constant speed v, and having an arbitrary functional form f. Another solution is given by

$$\psi(r, t) = \frac{g(r - vt)}{r}$$

and in this case the wave is converging toward the origin. The face that this expression blows up at $r = 0$ is of little practical concern.

A special case of the general solution

$$\psi(r, t) = C_1 \frac{f(r - vt)}{r} + C_2 \frac{g(r - vt)}{r} \quad ...(2.78)$$

is the *harmonic spherical wave*

$$\psi(r, t) = \left(\frac{A}{r}\right) \cos k(r \mp vt) \quad ...(2.79)$$

or
$$\psi(r, t) = \left(\frac{A}{r}\right) e^{ik(r \mp vt)} \quad ...(2.50)$$

wherein the constant A is called *the source strength*. At any fixed value of time, this represents a cluster of concentric spheres filling all space. Each wavefront, or surface of constant phase, is given by

$$kr = \text{constant}$$

Notice that the amplitude of any spherical wave is a function of r, where the term r^{-1} serves as an attenuation factor. Unlike the plane wave a spherical wave decreases in amplitude, thereby changing its profile, as it expands and moves out form the origin. Figure 2.19 illustrates this graphically by showing a "multiple exposure" of a spherical pulse at four different times. The pulse has the same extent in space at any point along any radius r, that is, the width of the pulse along the r-axis is a constant.

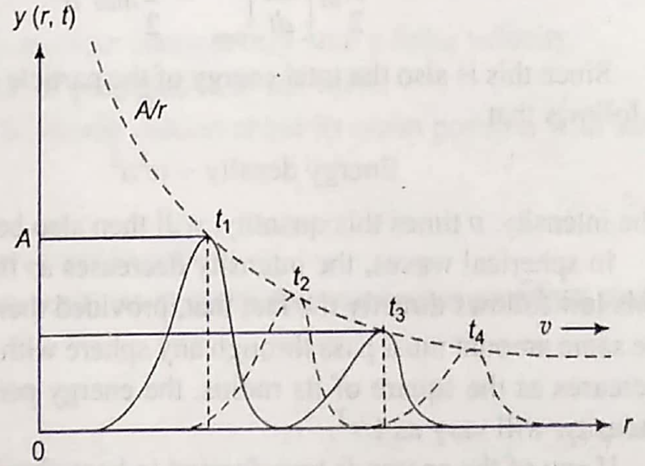

Fig. 2.19 *A"quadruple exposure" of a spherical pulse*

The outgoing spherical wave emanating from a point source and the incoming wave converging to a point are idealization. In actuality, light only approximates spherical waves, as it also only approximates plane waves.

As a spherical wavefront propagates out, its radius increases. Far enough away from the source, a small area of the wavefront will closely resemble a portion of a plane wave Fig. 2.20

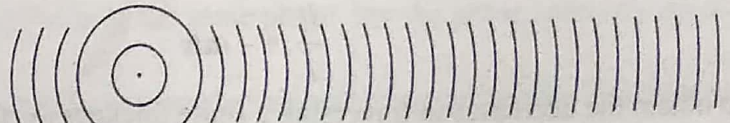

Fig. 2.20 *The flattening of spherical waves with distance.*

2.8 WAVE INTENSITY

Waves transport energy, and the *amount of it that flows per second across unit area perpendicular to the direction of travel is called the intensity of the wave.* If the wave flows continuously with the velocity v, there is a definite energy density, or total energy per unit volume. All the energy contained in a column of the medium of unit cross section and of length v will pass through the unit of area in 1 sec. Thus the intensity is given by the product of v and the energy density. Either the energy density or the intensity is proportional to the square of the amplitude and to the square of the frequency. To prove this proposition for sine waves in a medium, it is necessary only to determine the vibrational energy of a single particle executing simple harmonic motion.

The displacement of the particle executing simple harmonic motion is given by eqn.(2.6)

$$y = a \sin(\omega t - kx)$$

where k is the wave number and ω the angular frequency of particle oscillations. It is further expressed as

$$y = a \sin(\omega t - \alpha)$$

where α is the value of kx for that particle. The velocity of the particle is

$$\frac{dy}{dt} = \omega a \cos(\omega t - \alpha)$$

when $y = 0$, the sine vanishes and the cosine has its maximum value. Then the velocity becomes $-\omega a$, and the maximum kinetic energy

$$\frac{1}{2}m\left[\frac{dy}{dt}\right]^2_{max} = \frac{1}{2}m\omega^2 a^2$$

Since this is also the total energy of the particle and is proportional to the energy per unit volume, it follows that

$$\text{Energy density} \sim \omega^2 a^2$$

The intensity, v times this quantity, will then also be proportional to ω^2 and a^2. ...(2.81)

In spherical waves, the intensity decreases as the inverse square of the distance from the source. This law follows directly the fact that, provided there is no conversion of the energy into other forms, the same amount must pass through any sphere with the source as its center. Since the area of a sphere increases as the square of its radius, the energy per unit area at a distance r from the source, or the intensity, will vary as $1/r^2$.

If any of the energy is transformed to heat, that is to say, if there is *absorption*, the amplitude and intensity of plane waves will not be constant but decreases as the wave passes through the medium. Similarly with spherical waves, the loss of intensity will be more rapid than is required by the inverse-square law. For plane waves, the fraction dI/I of the intensity lost in traversing an infinitesimal thickness dx is proportional to dx, so that

$$\frac{dI}{I} = -\alpha dx$$

To obtain the decreases in traversing a finite thickness x, the equation is integrated to give

$$\int_0^x \frac{dI}{I} = -\alpha \int_0^x dx$$

Evaluating these definite integrals, we find

$$I_x = I_0 e^{-\alpha x} \quad ...(2.82)$$

This is called the exponential law of absorption. Fig. 2.21 is plot of the intensity against thickness according to this law for a medium having $\alpha = 0.4$ per cm. The wave equations may be modified to take account of absorption by multiplying the amplitude by the factor $e^{-\alpha x/2}$, since the amplitude varies with the square root of the intensity.

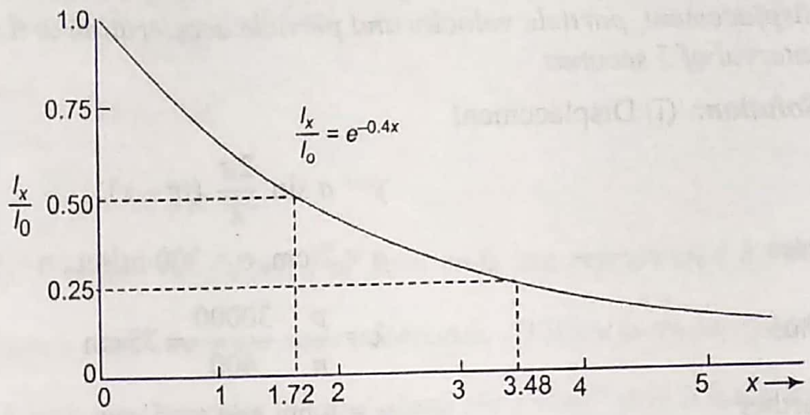

Fig. 2.21 *Decrease of intensity in an absorbing medium*

For light, the intensity can be expressed in ergs per square centimeter per second. Full sunlight, for example, has an intensity in these units of about 1.4×10^6. Here it is important to realize that not all this energy flux affects the eye, and not all that does is equally efficient. Hence the intensity as defined above does not necessarily correspond to the sensation of brightness, and it is more usual to find light flux expressed in visual units

2.9 DISTINCTION BETWEEN PROGRESSIVE AND STATIONARY WAVES

Progressive Waves:

1. This is an advancing wave which moves in the medium continuously with a finite velocity.
2. Energy flows across every plane in the direction of propagation of the wave.
3. Each particle of the medium executes simple harmonic motion about its mean position with same amplitude.
4. No particle of the wave is permanently at rest.
5. The phase of vibration varies continuously from point to point.
6. All the particles do not pass through their mean positions or reach their outermost positions simultaneously.

Stationary Waves:

1. There is no advancement of the wave in any direction.
2. There is no flow of energy across any plane.
3. Except nodes, all the particles of the medium execute simple harmonic motion with varying amplitudes.
4. Nodes are permanently at rest.
5. All the points between any pair of nodes vibrate in the same phase, but the phase suddenly reverses at each node.
6. All the particles pass through their mean positions or reach their outermost positions simultaneously twice in periodic time.