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Lattice as a Poset

A poset $(P, \leq)$ is called a lattice as a Poset if $\forall x, y \in P$, $\sup\{x,y\} \in P$ & $\inf\{x,y\} \in P$.

ex $P = \{1, 2, 3, 6, 9, 18, 27\}$
$x \leq y$ iff $x \mid y$.
P is not a lattice
$\because \sup\{18, 27\} = 54 \notin P$.

ex $(\mathbb{N}, \leq)$ is a lattice.
If $x, y \in \mathbb{N}$,
$\sup\{x,y\} = \max\{x,y\} \in \mathbb{N}$
& $\inf\{x,y\} = \min\{x,y\} \in \mathbb{N}$.

ex $A =$ any set.
$(\mathcal{P}(A), \subseteq)$ is a lattice
with $\inf\{X, Y\} = X \cap Y$
& $\sup\{X, Y\} = X \cup Y$.

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ex $G =$ any group.
$(\mathcal{P}(G), \subseteq)$ is a lattice with
$\inf\{X, Y\} = X \cap Y$
& $\sup\{X, Y\} = X \cup Y$.

Let Sub $G =$ Set of all Subgroups of G.
(Sub G, $\subseteq$) is a Poset.
But not a lattice
($\because$ Union of 2 Subgroups may not be a Subgroup of G).

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→ Every chain is a lattice as a Poset

Pf Let $(L, \leq)$ be a chain.

$\Rightarrow$ L is a poset

Let $x, y \in L$.

Since L is a chain,

Either $x \leq y$ or $y \leq x$.

Case 1 $x \leq y$

Then $\inf\{x, y\} = x \in L$

& $\sup\{x, y\} = y \in L$

Case 2 $y \leq x$

Then $\inf\{x, y\} = y \in L$

& $\sup\{x, y\} = x \in L$.

$\therefore$ L is a lattice as a Poset.

→ Let $(L, \leq)$ be a lattice ordered Set. Then TFAE

(i) $x \leq y$ in L

(ii) $\sup\{x, y\} = y$

(iii) $\inf\{x, y\} = x$

Defn 2 Lattice as an algebra.

A Set L with 2 binary Compositions $\wedge$ (meet) & $\vee$ (join) is Called a lattice if following properties hold in L:

i) $\wedge$ & $\vee$ are Commutative in L

ii) $\wedge$ & $\vee$ are associative in L

iii) Absorption laws hold

i.e. $a \wedge (a \vee b) = a$

& $a \vee (a \wedge b) = a$ $\forall a, b \in L$.

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→ If $(L, \wedge, \vee)$ is a lattice as an algebra

Then $\forall\, x, \cancel{y} \in L$, $x \wedge x = x$

& $x \vee x = x$

(Idempotent laws hold)

Pf $x \wedge x = \cancel{x \vee} \; x \wedge [x \vee (x \wedge x)]$

(By Absorption)

$= x$ (By absorption)

Similarly $x \vee x = x$.

<u>Defn 1 ⇒ Defn 2.</u>

Let $(L, \leq)$ be a lattice as a poset.

~~Then~~ Let $x, y \in L$

Define $x \wedge y = \inf\{x, y\} \in L$

& $x \vee y = \sup\{x, y\} \in L$

Then $\wedge$ & $\vee$ are Binary Comps in L.

1) $\wedge$ & $\vee$ are Commutative.

$x \wedge y = \inf\{x, y\} = \inf\{y, x\} = y \wedge x$

$x \vee y = \sup\{x, y\} = \sup\{y, x\} = y \vee x$.

2) $\wedge$ & $\vee$ are associative.

i) <u>Observe</u> $x \wedge y = \inf\{x, y\} \leq x$

& $x \wedge y \leq y$

& $\cancel{x \vee}$ $x \leq x \vee y$, $y \leq x \vee y$.

ii) ~~Obsa~~: Isotone laws hold in L

If $x \leq y$ in L, $x \wedge z \leq y \wedge z$

& $x \vee z \leq y \vee z \;\; \forall z \in L$.

(Prove)

<u>To prove</u> $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.

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$x \wedge (y \wedge z) \leq y \wedge z$
$\leq z$

$\Rightarrow$ By transitivity $x \wedge (y \wedge z) \leq z$ — (1)
~~& x ∧ (y∧z)~~

Also $y \wedge z \leq y$
$\Rightarrow x \wedge (y \wedge z) \leq x \wedge y$ (isotone law) — (2)

By (1) & (2) $x \wedge (y \wedge z)$ is a l.b of $x \wedge y$ & z
& lb $\leq$ glb
$\Rightarrow \boxed{x \wedge (y \wedge z) \leq (x \wedge y) \wedge z}$

Now $(x \wedge y) \wedge z \leq x \wedge y \leq x$
~~& x ∧ y ≤ x~~
$\Rightarrow (x \wedge y) \wedge z \leq x$ — (3)

Also $x \wedge y \leq y$
$\Rightarrow (x \wedge y) \wedge z \leq y \wedge z$ (Isotone law) — (4)

By (3) & (4), $(x \wedge y) \wedge z$ is a l.b of x & $y \wedge z$
lb $\leq$ glb
$\Rightarrow \boxed{(x \wedge y) \wedge z \leq x \wedge (y \wedge z)}$

Hence By Antisymmetry
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$

Similarly prove $(x \vee y) \vee z = x \vee (y \vee z)$

3) Absorption laws hold.

To prove $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$.

$\boxed{x \wedge (x \vee y) \leq x}$. Also $x \leq x$ & $x \leq x \vee y$

$\Rightarrow x$ is ~~an~~ a l.b of x & $x \vee y$.

$\Rightarrow \boxed{x \leq x \wedge (x \vee y)}$

Hence $x = x \wedge (x \vee y)$. Absorption holds

$\therefore$ L_6 is a lattice as an algebra