(2) Two oscillations of slightly different frequencies moving in the same direction (Beats).
(3) Two oscillations of the same frequency moving in the opposite directions (standing waves).
(4) Two oscillations of same frequency but perpendicular to each other (Lissajous Figures).

1.4 SUPERPOSITION OF TWO COLLINEAR HARMONIC OSCILLATIONS OF SAME FREQUENCIES

Let us consider the case of two simple harmonic motions of the same period i.e. same frequency moving in the same positive direction of x-axis with amplitudes a_1 and a_2 and phase constants α_1 and α_2. The displacements of two such harmonic motions are given by

$$x_1 = a_1 \cos(\omega t + \alpha_1) \tag{1.29}$$

$$x_2 = a_2 \cos(\omega t + \alpha_2) \tag{1.30}$$

The resultant motion of the system, moving in the x-direction under the simultaneous effect of the two harmonic oscillations, can be obtained analytically as follows.

Mathematical analysis: Making use of principle of superposition, according to which the resultant displacement is equal to the sum of individual displacements, we get

$$x = x_1 + x_2$$

or

$$x = a_1 \cos(\omega t + \alpha_1) + a_2 (\cos \omega t + \alpha_2) \tag{1.31}$$

From a trigonometric expansion,

$$\cos(\phi_1 + \phi_2) = \cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2$$

We can write the resultant displacement as

$$x = a_1 (\cos \omega t \cos \alpha_1 - \sin \omega t \sin\alpha_1) + a_2 (\cos \omega t \cos\alpha_2 - \sin \omega t \sin \alpha_2) \tag{1.32}$$

or

$$x = (a_1 \cos \alpha_1 + a_2 \cos \alpha_2)\cos \omega t - (a_1 \sin \alpha_1 + a_2 \sin \alpha_2)\sin \omega t \tag{1.33}$$

Let

$$a_1 \cos \alpha_1 + a_2 \cos \alpha_2 = A \cos \theta \tag{1.34}$$

and

$$a_1 \sin \alpha_1 + a \sin \alpha_2 = A \sin \theta$$

where A and θ are constants to be determined. Putting these values in eqn. (1.32), we get

$$x = A \cos \omega t \cos \theta - A \sin \omega t \sin \theta \tag{1.35}$$

or

$$x = A \cos(\omega t + \theta)$$

This equation shows that the resultant displacement is also a simple harmonic motion of amplitude A and a phase constant θ. The angular frequency of the resulting motion is the same as that of individual harmonic oscillations. The values of A and θ can be determined from eqns. (1.33) and (1.34). Squaring and adding eqns. (1.33) and (1.34), we get

$$A^2(\cos^2\theta + \sin^2\theta) = (a_1 \cos\alpha_1 + a_2 \cos\alpha_2)^2 + (a_1 \sin\alpha_1 + a_2 \sin\alpha_2)^2$$

$$A^2 = a_1^2 \cos^2\alpha_1 + a_2^2 \cos^2\alpha_2 + 2a_1a_2 \cos\alpha_1 \cos\alpha_2 + a_1^2 \sin^2\alpha_1$$

$$+ a_2^2 \sin^2\alpha_2 + 2a_1a_2 \sin\alpha_1 \sin\alpha_2$$

or

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2(\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2)$$

or

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\alpha_2 - \alpha_1)$$

or

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos\alpha \quad (1.36)$$

where $\alpha = \alpha_2 - \alpha_1$ is a constant phase and A the resultant amplitude at any point depends on the phase difference of the two waves meeting at the point.

The value of θ can be obtained by dividing equation (1.34) by eqn. (1.33). Thus

$$\tan\theta = \frac{a \sin\alpha_1 + a_2 \sin\alpha_2}{a \cos\alpha_1 + a_2 \cos\alpha_2} \quad (1.37)$$

Thus we see that the resultant of superposition of two collinear simple harmonic oscillations of same frequency is a simple harmonic oscillation having an amplitude given by eqn. (1.36) and a constant phase, eqn. (1.37). The resultant oscillation has the same frequency ω.

Special cases:

(1) *Superposition of two waves in the same phase, i.e.,* $\alpha = 0$.

The resultant amplitude in this case is given by

$$A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos 0)^{1/2} \quad \text{[From eq. (1.36)]}$$

$$= (a_1^2 + a_2^2 + 2a_1 a_2)^{1/2}$$

or

$$A = (a_1 + a_2) \quad ...(1.38)$$

(2) *Superposition of two waves in the opposite phase, i.e.,* $\alpha = \pi$.

The resultant amplitude in this case is given by

$$A = (a_1^2 + a_2^2 + 2a_1 a_2 \cos\pi)^{1/2}$$

$$= (a_1^2 + a_2^2 - 2a_1 a_2)^{1/2} \quad [\because \cos\pi = -1]$$

or

$$A = (a_1 - a_2) \quad ...(1.39)$$

Thus if the two sound waves or light waves of same frequency and having amplitudes a_1 and a_2 are travelling along the same line reach a certain point exactly in phase, then the resultant amplitude at this point would be $(a_1 + a_2)$ and hence a loud sound or maximum intensity in case of light waves. But, if they arrive in opposite phase, the resultant amplitude will the $(a_1 - a_2)$ and hence a feeble sound. For intermediate phase difference, the resultant amplitude lies between $(a_1 + a_2)$ and $(a_1 - a_2)$.

Graphical Method: Vector addition of simple harmonic motion gives a simple graphical method of getting the resultant of two simple harmonic motions of equal frequencies. In Fig. (1.6a), let OP_1 be a vector of length a_1, making an angle of $(\omega t + \alpha_1)$ with the x-axis at time t, where α_1 is phase constant and ω is the angular frequency of the oscillations. The projection ON_1 of OP, on the x-axis is the displacement x_1, of this motion at time t. Let OP_2 be a vector of length a_2 making an angle $(\omega t + \alpha_2)$

with x-axis. Its projection ON_2 on x-axis is second harmonic motion of same frequency ω, amplitude a_2 and phase constant α_2. The superposition of two motions is given by the vector OP which can be obtained by parallelogram law of vector sum of OP_1 and P_1P as shown in Fig (1.6b), the vector P_1P is equal to vector OP_2. Since $\underline{|P_1ON_1} = \omega t + \alpha_1$ and $\underline{|P_2ON_2} = \omega t + \alpha_2$, the angle between OP_1 and P_1P is just $\alpha_2 - \alpha_1 = \alpha$. Thus, we have from rt. angle triangle POQ,

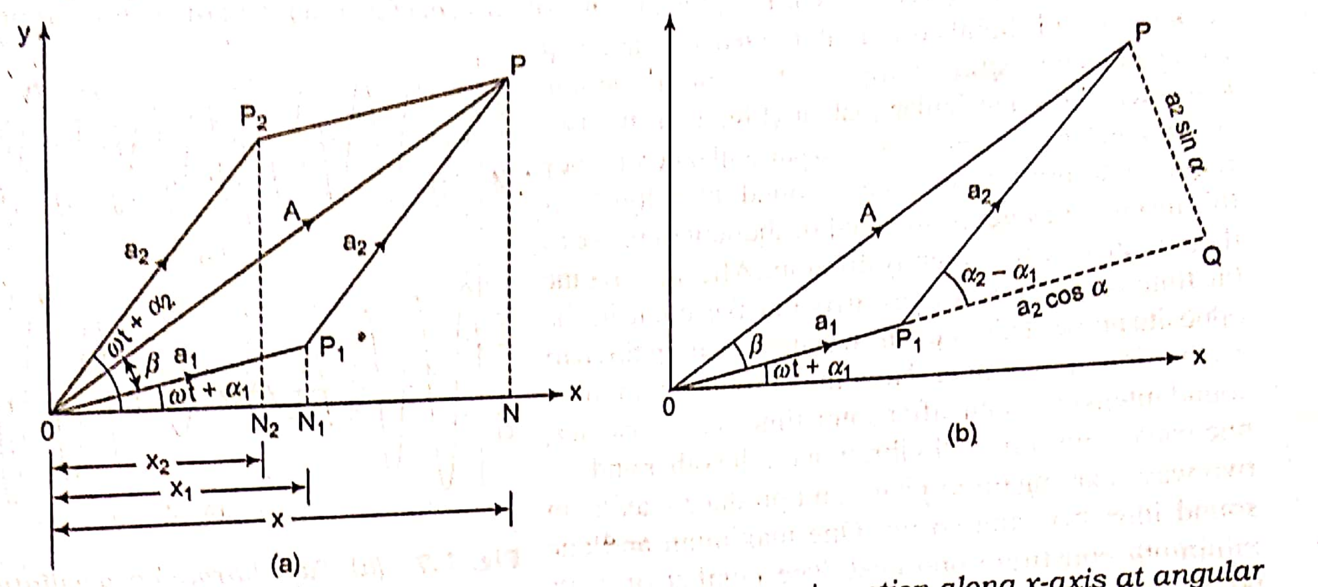

Fig. 1.6. *Addition of vectors, each representing simple harmonic motion along x-axis at angular frequency ω to give a resulting SHM displacement*

$$A^2 = (a_1 + a_2 \cos \alpha)^2 + (a_2 \sin \alpha)^2$$

or

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos \alpha$$

which is same as obtained analytically (eqn. 1.36)

The total phase of resultant motion is given by $\underline{|PON}$ and let this is equal to $(\omega t + \delta)$ where δ is the phase constant of the resultant motion. From Fig. (1.6b), we have

$$\delta = \beta + \alpha_1$$

$\therefore$

$$\tan \delta = \tan (\beta + \alpha_1) = \frac{\tan \beta + \tan \alpha_1}{1 + \tan \beta \tan \alpha_1}$$

now

$$\tan \beta = \frac{a_2 \sin \alpha}{a_1 + a_2 \cos \alpha} \qquad \text{(where } \alpha = \alpha_2 - \alpha_1\text{)}$$

Putting for $\tan \beta$ in the above equation and simplifying, we get

$$\tan \delta = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2}$$

which is same as eqn. (1.37) obtained analytically.

1.5 SUPERPOSITION OF TWO SIMPLE HARMONIC OSCILLATIONS OF DIFFERENT FREQUENCIES: BEATS

When two simple harmonic waves of slightly differing frequencies (*e.g.*, from two tuning forks