The Cauchy - Euler Equation :-

The equation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x) \quad \text{--- (1)}$$

is called Cauchy - Euler equation. where a_0, $a_1, a_2, \ldots, a_{n-1}, a_n$ are constants.

The equation (1) can be reduced to a linear differential equation with constant coefficients by using the transformation $x = e^t$ or $t = \ln x$. And we know how to solve linear differential equation with constant coefficients.

Example :- Solve $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^3$ --- (1)

Let $x = e^t$. or $t = \ln x \Rightarrow \frac{dt}{dx} = \frac{1}{x}$

then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{1}{x}\frac{d^2y}{dt^2}\cdot\frac{dt}{dx} - \frac{1}{x^2}\frac{dy}{dt}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}}$$

~~Substitute~~ Substitute the value of $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$, in ①.

$$x^2 \times \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 2x\cdot\frac{1}{x}\frac{dy}{dt} + 2y = e^{3t}$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = e^{3t}$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t} \quad \text{— (2)}$$

Auxiliary equation of ② is

$$m^2 - 3m + 2 = 0$$

$$m^2 - 2m - m + 2 = 0$$

$$m(m-2) - 1(m-2) = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

Complementary function of (2)

$$\boxed{y_c = C_1 e^{t} + C_2 e^{2t}}$$

P.I. corresponding to e^{3t}

$$= \frac{1}{D'^2 - 3D' + 2} e^{3t}$$

$$\boxed{D' = \frac{d}{dt}}$$

$$= \frac{1}{9 - 9 + 2} e^{3t}$$

$$y_p = \frac{1}{2} e^{3t}$$

then general soln of (2)

$$y = y_c + y_p$$

$$\boxed{y = C_1 e^{t} + C_2 e^{2t} + \frac{1}{2} e^{3t}}$$

General soln of (1) is

$$\boxed{y = C_1 x + C_2 x^2 + \frac{1}{2} x^3}$$

Exercise:-

$$x^3\frac{d^3y}{dx^3} - 4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} - 8y = 4\ln x$$

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$$x^3 \frac{d^3y}{dx^3} - 4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} - 8y = 4 \ln x$$

$x = e^t$ or $t = \ln x$.

$$\frac{dt}{dx} = \frac{1}{x}$$

$$x \frac{dy}{dx} = \frac{dy}{dt} \Rightarrow \boxed{x Dy = D'y}$$

where, $D = \frac{d}{dx}$

& $D' = \frac{d}{dt}$

&

$$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

or

$$x^2 D^2 y = D'^2 y - D'y$$

$$\boxed{x^2 D^2 y = D'(D'-1)y}$$

Similarly

$$x^3 D^3 y = D'(D'-1)(D'-2)y$$

$$\boxed{x^n D^n y = D'(D'-1)(D'-2) \cdots (D'-n+1)y}$$

where $D' = \frac{d}{dt}$

& $D = \frac{d}{dy}$.

$dx^2 \qquad dt^2 \qquad dt^2 \qquad dt$

Note that this is precisely the relation we found in Example 4.44 and stated above.

Exercises

Find the general solution of each of the differential equations in Exercises 1–19. In each case assume $x > 0$.

1. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 3y = 0.$

2. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - 4y = 0.$

3. $4x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 3y = 0.$

4. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 4y = 0.$

5. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + 4y = 0.$

6. $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + 13y = 0.$

7. $3x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 2y = 0.$

8. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + 9y = 0.$

9. $9x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = 0.$

10. $x^2 \dfrac{d^2y}{dx^2} - 5x \dfrac{dy}{dx} + 10y = 0.$

11. $x^3 \dfrac{d^3y}{dx^3} - 3x^2 \dfrac{d^2y}{dx^2} + 6x \dfrac{dy}{dx} - 6y = 0.$

12. $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} - 10x \dfrac{dy}{dx} - 8y = 0.$

13. $x^3 \dfrac{d^3y}{dx^3} - x^2 \dfrac{d^2y}{dx^2} - 6x \dfrac{dy}{dx} + 18y = 0.$

14. $x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 6y = 4x - 6.$

15. $x^2 \dfrac{d^2y}{dx^2} - 5x \dfrac{dy}{dx} + 8y = 2x^3.$

16. $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = 4 \ln x.$

17. $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + 4y = 2x \ln x.$

18. $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 4 \sin \ln x.$

19. $x^3 \frac{d^3y}{dx^3} - x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = x^3.$

Solve the initial-value problem in each of Exercises 20–27. In each case assume $x > 0$.

20. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 10y = 0, \quad y(1) = 5, \quad y'(1) = 4.$

21. $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 6y = 0, \quad y(2) = 0, \quad y'(2) = 4.$

22. $x^2 \frac{d^2y}{dx^2} + 5x \frac{dy}{dx} + 3y = 0, \quad y(1) = 1, \quad y'(1) = -5.$

23. $x^2 \frac{d^2y}{dx^2} - 2y = 4x - 8, \quad y(1) = 4, \quad y'(1) = -1.$

24. $x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 4y = 4x^2 - 6x^3, \quad y(2) = 4, \quad y'(2) = -1.$

25. $x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 6y = 10x^2, \quad y(1) = 1, \quad y'(1) = -6.$

26. $x^2 \frac{d^2y}{dx^2} - 5x \frac{dy}{dx} + 8y = 2x^3, \quad y(2) = 0, \quad y'(2) = -8.$

27. $x^2 \frac{d^2y}{dx^2} - 6y = \ln x, \quad y(1) = \frac{1}{6}, \quad y'(1) = -\frac{1}{6}.$

28. Solve:

$$(x + 2)^2 \frac{d^2y}{dx^2} - (x + 2) \frac{dy}{dx} - 3y = 0.$$

29. Solve:

$$(2x - 3)^2 \frac{d^2y}{dx^2} - 6(2x - 3) \frac{dy}{dx} + 12y = 0.$$

4.6 STATEMENTS AND PROOFS OF THEOREMS ON THE SECOND HOMOGENEOUS LINEAR EQUATION