Fourier Series

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \quad \text{--- (1)}$$

To find

a_0: Integrating (1) w.r.t x b/w the limits $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\, dx$$

$$= a_0 \cdot 2\pi + 0 + 0$$

$$\therefore \boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx}$$

[interval may be $c < x < c+2\pi$ ie $c, c+2\pi$]

a_n: (1) $\times \cos nx$ & integrate w.r.t x b/w the limits $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x) \cos nx\, dx = a_0 \int_{-\pi}^{\pi} \cos nx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos^2 nx\, dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \cos nx \sin nx\, dx$$

$$= a_0 \cdot 0 + a_n \cdot \pi + 0$$

$$\therefore \boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx} \quad n = 1, 2, 3 \dots$$

b_n: (1) $\times \sin nx$ & integrate w.r.t x b/w the limits $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \int_{-\pi}^{\pi} \sin nx\, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin nx\, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin^2 nx\, dx$$

$$= a_0 \cdot 0 + a_n \cdot 0 + b_n \cdot \pi$$

$$\therefore \boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx} \quad n = 1, 2, 3 \dots$$

Half Range Series:

cosine series: If $\boxed{f(x) = f(-x)}$ ie Even function

then $\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi} \int_{0}^{\pi} f(x)dx}$

& $\boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx}$

then (1) reduces to $f(x) = \boxed{a_0 + \sum_{n=1}^{\infty} a_n \cos nx}$

It is called fourier cosine series in interval $(0, \pi)$

Sine series: if $\boxed{f(-x) = -f(x) \text{ ie odd function}}$ then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx}$$

so (I) reduces to

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

Note 1: Here a_0, a_n, b_n $\rightarrow$ $2\times$ a_0, a_n, b_n with limit 0 to π

Exponential form / complex form

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (I)}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{e^{inx} + e^{-inx}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{inx} - e^{-inx}}{2i}\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[\frac{a_n - ib_n}{2}\right] e^{inx} + \sum_{n=1}^{\infty} \left[\frac{a_n + ib_n}{2}\right] e^{-inx}$$

$$= c_0 e^{i0x} + \sum_{n=1}^{\infty} c_n e^{inx} + \sum_{n=1}^{\infty} c_{-n} e^{-inx} \quad \text{--- (II)}$$

$$\boxed{f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}}$$ of is expo form / complex form --- (III)

Where $c_0 = a_0$

$c_n = (a_n - ib_n)/2$

$c_{-n} = (a_n + ib_n)/2$

Evaluation of c_n: multiply (III) by e^{-imx} & integrate w.r.t x b/w limits $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx = \sum_{n=-\infty}^{+\infty} c_n \int_{-\pi}^{\pi} e^{-imx} \cdot e^{inx}\,dx$$

$$= \sum_{n=-\infty}^{+\infty} c_n \int e^{i(n-m)x}\,dx, \text{ when } n = m \text{ then}$$

$$= c_m \int_{-\pi}^{\pi} dx = c_m \cdot 2\pi \quad \text{ie } c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-imx}\,dx$$

or $\boxed{c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx}$ & $\boxed{c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{inx}\,dx}$

Change of interval of Fourier series (period 0 to 2π)

some time interval changes $(-\pi, \pi)$ to $(-l, l)$

introduce a variable $y = \frac{\pi x}{l}$ or $x = \frac{ly}{\pi}$

$\because$ $2l$ is the interval for variable x
$\therefore$ 1 " " " $\frac{x}{2l}$
& 2π " " " $\frac{x}{2l} \times 2\pi = \frac{x\pi}{l} = y$ (say)

$$\therefore f(x) = f\left(\frac{ly}{\pi}\right) = \psi(y) \text{ (say)}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos ny + \sum_{n=1}^{\infty} b_n \sin ny$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{--- (1)}$$

$$\therefore a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(y)\,dy = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{ly}{\pi}\right) dy$$

$$= \frac{1}{2\pi}\int_{-l}^{l} f(x)\,\frac{\pi}{l}\,dx$$

$$= \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

put $y = \frac{\pi x}{l}$
$\therefore dy = \frac{\pi}{l}dx$
when $y \to \pi$ then
$x = \frac{ly}{\pi} = \frac{l \cdot \pi}{\pi}$
$\therefore x = l$

$$\therefore \boxed{a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx}$$

Similarly

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(y) \cos ny\, dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{ly}{\pi}\right) \cos \frac{n\pi x}{l} \cdot d\left(\frac{\pi x}{l}\right)$$

$$= \frac{1}{\pi}\int_{-l}^{+l} f(x) \cos \frac{n\pi x}{l} \cdot \left(\frac{\pi}{l}\right) dx$$

put $y = \frac{\pi x}{l}$
$dy = \frac{\pi}{l}dx$

$$\boxed{a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos \frac{n\pi x}{l}\, dx}$$

$$\& \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(y) \sin ny\, dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{ly}{\pi}\right) \sin \frac{n\pi x}{l} \cdot d\left(\frac{\pi x}{l}\right)$$

$$= \frac{1}{\pi}\int_{-l}^{+l} f(x) \sin \frac{n\pi x}{l} \cdot \left(\frac{\pi}{l}\right) dx$$

$$\boxed{b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l}\, dx}$$

Note!
$f(x) = x, \; 0 < x < \pi/2$
$= \pi/2, \; \frac{\pi}{2} < x < \pi$
Here $2l = \pi$
$\therefore l = \pi/2$.

Half range series

Cosine series:

$$\boxed{\begin{aligned} a_0 &= \frac{1}{l}\int_0^l f(x)\,dx \\ a_n &= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \end{aligned}}$$

$$\therefore \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}}$$

Sine series:

$$\boxed{b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx}$$

$$\therefore \boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}}$$

Parseval's Identity / formula

$$\boxed{\int_{-l}^{+l}[f(x)]^2dx = l\left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]} \quad \text{— (1)}$$

Assume that the Fourier series corresponding to $f(x)$ converges uniformly to $f(x)$ in $(-l, l)$. when the integral (1) exists

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\,dx \quad \text{— (2)}$$

(2) $\times f(x)$ & integrate w.r.t x b/w the limits $-l$ to l

$$\therefore \int_{-l}^{+l}[f(x)]^2dx = a_0\int_{-l}^{+l} f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n \int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}\,dx + b_n\int_{-l}^{+l} f(x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= a_0[a_0\cdot 2l] + \sum_{n=1}^{\infty} a_n[a_n l] + \sum_{n=1}^{\infty} b_n[b_n l]$$

$$\int_{-l}^{+l}[f(x)]^2dx = l\left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2+b_n^2\right]$$

Since :-

$$\left.\begin{aligned} \int_{-l}^{l} f(x)\,dx &= 2la_0 \\ \int_{-l}^{+l} f(x)\cos\frac{n\pi x}{l}\,dx &= la_n \\ \int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx &= lb_n \end{aligned}\right\}$$ as done in change of interval

Note : if

$\boxed{0<x<2l}$ then

$$\boxed{\int_0^{2l} [f(x)]^2 dx = l\left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2\right]}$$

Half Range Series :

Cosine Series $(0<x<l)$: $$\boxed{\int_0^{l} [f(x)]^2 dx = \frac{l}{2}\left[2a_0^2 + \sum_{n=1}^{\infty} a_n^2\right]}$$

Sine Series $(0<x<l)$: $$\boxed{\int_0^{l} [f(x)]^2 dx = \frac{l}{2}\left[\sum_{n=1}^{\infty} b_n^2\right]}$$

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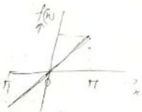
Que Represent the function $f(x) = x \; ; \; -\pi < x < \pi$ in the form of a fourier series & deduce that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = \frac{\pi}{4}$$



Sol :- The fourier series expansion of the function $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)\Big|_0^{\pi} - \int_0^{\pi} 1\cdot\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi(-\cos n\pi)}{n} + \left(\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}(-\cos n\pi) + 0 + 0\right]$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$\therefore x = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} +\frac{2}{n}(-1)^{n+1}\sin nx \qquad \text{put } x = \frac{\pi}{2}$$

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

$$\boxed{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots}$$

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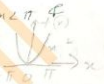
Que :- Represent the fourier series expansion of x^2, $-\pi < x < \pi$ & deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$



Sol :- the fourier series expansion of $f(x) = x^2$ be

$$f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ---(1)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\int_{0}^{\pi} x^2 dx = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}$
$= \frac{\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(x^2 \frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n}\,dx\right] = \frac{2}{\pi}\left[0 - \frac{2}{n}\int_0^{\pi} x \sin nx\,dx\right]$$

$$= \frac{2}{\pi}\cdot(-1)\frac{2}{n}\int_0^{\pi} \overset{*}{x \sin nx}\,dx = \frac{2}{n}\cdot\frac{2}{n}(-1)^{n} = \frac{4}{n^2}(-1)^{n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\,dx = 0 \quad [\because \text{function is odd}]$$

$$\therefore f(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^{n} \cos nx$$

put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Hence $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$

$*$ $\int_0^{\pi} x \sin nx\,dx = \frac{-\pi}{n}\cos n\pi$

see prev. problem.

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Que :- Expand $f(x) = \sin x$; $0 < x < \pi$ in fourier Cosine series.

Sol :- fourier Cosine series is

$$f(x) = \sin x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi}$

$$= \frac{1}{\pi}\left[1+1\right] = \frac{2}{\pi}$$

& $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{\cos(n+1)\pi - 1}{n+1} - \frac{\cos(n-1)\pi - 1}{n-1}\right]$$

$$= -\frac{1}{\pi}\left[\frac{(-1)^{n+1} - 1}{n+1} - \frac{(-1)^{n-1} - 1}{n-1}\right]$$

$$= -\frac{2}{\pi}\left[\frac{(-1)^n + 1}{n^2 - 1}\right]$$

$$= \begin{cases} 0 & \text{for } n = \text{odd} \\ -\frac{4}{\pi}\cdot\frac{1}{(n^2-1)} & \text{for } n = \text{even} \end{cases}$$

$$\therefore \sin x = a_0 + \sum_{n=2,4,6\ldots}^{\infty} a_n \cos nx$$

$$\Rightarrow \sin x = \frac{2}{\pi} + \left(-\frac{4}{\pi}\right)\sum_{n=2,4,6\ldots} \frac{\cos nx}{n^2-1}$$

$$\Rightarrow \boxed{\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right]}$$

Qn:- Expand $f(x) = \cos 2x$ in a series of sines in the interval

Sol:- Fourier sine series of $f(x) = \cos 2x$ be

$$f(x) = \cos 2x = \sum_{n=1}^{\infty} b_n \sin nx$$

Where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos 2x \sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+2)x + \sin(n-2)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+2)x}{n+2} + \frac{\cos(n-2)x}{n-2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos(n+2)\pi - 1}{n+2} + \frac{\cos(n-2)\pi - 1}{n-2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^{n+2} - 1}{n+2} + \frac{(-1)^{n-2} - 1}{n-2}\right]$$

$$= \frac{1}{\pi}\cdot\frac{2n}{n^2-4}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0 & \text{when } n = \text{even} \\ \dfrac{4}{\pi(n^2-4)} & \text{when } n = \text{odd} \end{cases}$$

$$\therefore\ f(x) = \cos 2x = \sum_{n=\text{odd (ie } 1,3,5,\ldots)}^{\infty} b_n \sin nx$$

$$\Rightarrow \cos 2x = \frac{4}{\pi}\sum_{n=1,3,5,7\ldots}\frac{n}{n^2-4}\sin nx$$

$$\Rightarrow \boxed{\cos 2x = \frac{4}{\pi}\left[-\frac{\sin x}{3} + \frac{3}{5}\sin 3x + \frac{5}{21}\sin 5x + \cdots\right]}$$

Find the complex fourier series of $f(x) = e^x$; $-\pi < x < \pi$ and $f(x+2\pi) = f(x)$

Sol:- complex series of $f(x) = e^x$ be

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx} \quad \text{--------- (1)}$$

Now
$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{2\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$c_0 = \frac{1}{2\pi}\left[e^{\pi} - e^{-\pi}\right] = \frac{1}{\pi}\sinh\pi$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\cdot e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-in)}\,dx = \frac{1}{2\pi}\left[\frac{e^{x(1-in)}}{1-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{\pi(1-in)} - e^{-\pi(1-in)}}{(1-in)}\right]$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(1-in)}\left[(e^{\pi}\cdot e^{-in\pi}) - e^{-\pi}\cdot e^{+in\pi}\right]$$

$$= \frac{1}{2\pi}\cdot\frac{1}{1-in}\left[e^{\pi}\cdot\{\cos n\pi - i\underset{0}{\sin n\pi}\} - e^{-\pi}\{\cos n\pi + i\underset{0}{\sin n\pi}\}\right]$$

$$= \frac{1}{2\pi}\cdot\frac{1}{1-in}\left[e^{\pi}(-1)^n - e^{-\pi}(-1)^n\right]$$

$$= \frac{1}{2\pi}\frac{(-1)^n}{1-in}\left[e^{\pi} - e^{-\pi}\right] = \frac{(-1)^n \times}{\pi(1-in)\times}\sinh\pi$$

$$= \frac{(-1)^n(1+in)}{1+n^2}\cdot\frac{\sinh\pi}{\pi}$$

$\therefore$ (1) becomes
$$\boxed{e^x = \frac{\sinh\pi}{\pi}\sum_{n=-\infty}^{+\infty}\frac{(-1)^n(1+in)e^{inx}}{1+n^2}}$$

$$= \frac{\sinh\pi}{\pi} + \frac{\sinh\pi}{\pi}\Big[\sum_{n=}$$

Que:- Expand for $f(x) = K$; for $0 < x < 2$ in a half range series of (i) Sine series
(ii) Cosine series

Sol:- (i) Sine Series : $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$

where $b_n = \frac{2}{l} \int_{0}^{+l} f(x) \sin \frac{n\pi x}{l} \, dx$

$$= \frac{2}{2} \int_0^2 K \sin \frac{n\pi x}{2} \, dx = \frac{K}{1} \left[\frac{-\cos \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right]_0^2$$

$$= \frac{-K \cdot 2}{n\pi} \left[\cos \frac{2n\pi}{2} - \cos 0 \right]$$

$$= \frac{-2K}{n\pi} \left[(-1)^n - 1 \right] = \frac{2K}{n\pi} \left[1 - (-1)^n \right]$$

$$= \begin{cases} 0 & \text{for } n = \text{even} \\ \frac{4K}{n\pi} & \text{for } n = \text{odd} \end{cases}$$

$$\therefore f(x) = \frac{2K}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{2}$$

$$\boxed{K = \frac{4K}{\pi} \left[\sin \frac{\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} + \frac{1}{5} \sin \frac{5\pi x}{2} + \cdots \right]}$$

(ii) Cosine Series : $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

where $a_0 = \frac{1}{l} \int_0^l f(x) dx = \frac{1}{2} \int_0^2 K \, dx = \frac{1}{2} K (x)_0^2$

$$= K$$

$$a_n = \frac{2}{2} \int_0^2 K \cos \frac{n\pi x}{2} \, dx = K \left[\frac{\sin \frac{n\pi x}{2}}{\frac{n\pi}{2}} \right]_0^2$$

$$= \frac{K \, 2}{n\pi} \left[\sin \frac{2}{2} n\pi - \sin 0 \right]$$

$$= \frac{2K}{n\pi} (\sin n\pi) = 0$$

$$f(x) = K + \sum_{n=1}^{\infty} \frac{2K}{n\pi} \sin n\pi \cos \frac{n\pi x}{2}$$