3.5 Conduction of Heat in Solids :-

Let us consider a domain D^* bounded by a closed surface B^*.

Let $u(x, y, z, t)$ be the temperature at any point (x, y, z) at any time t.

if temperature is not constant, we know heat flows from higher temp. to lower temp.

Also by Fourier's law : rate of flow is proportional to the gradient of the temp.

$\Rightarrow$ the velocity of body can be written as

$$v = -K(\text{grad}(u)) \quad \Big\} \because v \propto \text{grad}(u)$$
$$= -K \nabla u$$

where K is a constant,
it is called thermal conductivity of the body.

Now, let D be any arbitrary ~~body~~ domain bounded by closed surface B in D^*

$\Rightarrow$ amount of heat leaving D per unit time is

$$\iint_B v_n \, dS$$

where $v_n = v \cdot n$

i.e component of v in the direction of outer normal unit ~~n of~~ vector n of B

$\Rightarrow$ By Gauss theorem (Divergence theorem)

$$\iint_B v_n \, ds = \iiint_D \operatorname{div}(-K \operatorname{grad}(u)) \, dx\,dy\,dz$$

$$= -K \iiint_D \nabla^2 u \, dx\,dy\,dz$$

also the amount of heat in D is

$$\iiint_D \sigma \rho u \, dx\,dy\,dz$$

where ρ is density of the material, σ is specific heat,

Assuming integration & differentiation ~~are~~ can be interchange.

$\Rightarrow$ the rate of heat decrease in D is

$$-\iiint_D \sigma \rho \frac{\partial u}{\partial t} \, dx\,dy\,dz$$

$\therefore$ rate of decrease of heat in D is equal to the amount of heat leaving D per unit time.

$$\Rightarrow -\iiint_D \sigma \rho u_t \, dx\,dy\,dz = -K \iiint_D \nabla^2 u \, dx\,dy\,dz$$

$$\Rightarrow -\iiint \left[\sigma \rho u_t - K \nabla^2 u\right] dx\,dy\,dz = 0$$

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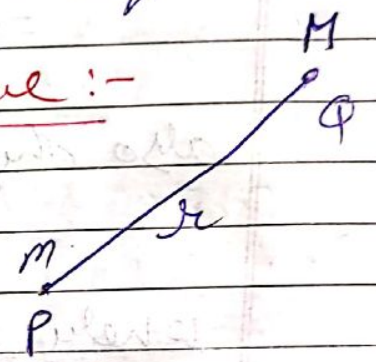
$\because$ integral is zero in D (arbitrary)

$\Rightarrow$ integrand is zero

$\Rightarrow$
$$\boxed{u_t = k\nabla^2 u} \quad \text{— (1)}$$

where $k = K/\rho\sigma$

this is known as heat equation.



* The Gravitational Potential :-

We consider two particles of mass m & M, at P & Q

let r be the distance b/w P & Q

$\Rightarrow$ By Newton's law of gravitation

$$F = \frac{GmM}{r^2}$$; where G is the gravitational constant

Assume $G = 1$

then $F = \dfrac{mM}{r^2}$

$\vec{r}$ represents the vector PQ

then

Force at Q due to P $\leftarrow$ $$F = \frac{-m\vec{r}}{r^3} = \nabla\left(\frac{m}{r}\right)$$

This is called the intensity of the gravitational field of force.

Now, we suppose that a particle of unit mass moves under the attraction of the particle of mass m at P from infinity upto Q.

Then the total work done by the force F is

$$\int_{\infty}^{r} F\,dr = \int_{\infty}^{r} \nabla\left(\frac{m}{r}\right) dr$$

$$= \frac{m}{r}$$

This is called the potential at Q due to P

it is denoted as $\boxed{V = -\frac{m}{r}}$

-ve sign is for negative to pull the body at Q by applying force in opposite direction}

$\Rightarrow$

$$F = -\nabla V$$

$\Rightarrow$ we we consider the masses $m_1, m_2 \ldots m_n$, whose distance from Q are $r_1, r_2, \ldots, r_n$ resp. then force at Q due to system is

$$F = \sum_{K=1}^{n} \nabla\left(\frac{m_K}{r_K}\right) = \nabla \sum_{K=1}^{n} \frac{m_K}{r_K}$$

$\Rightarrow$ total work done by the force acting on the particle of unit mass is

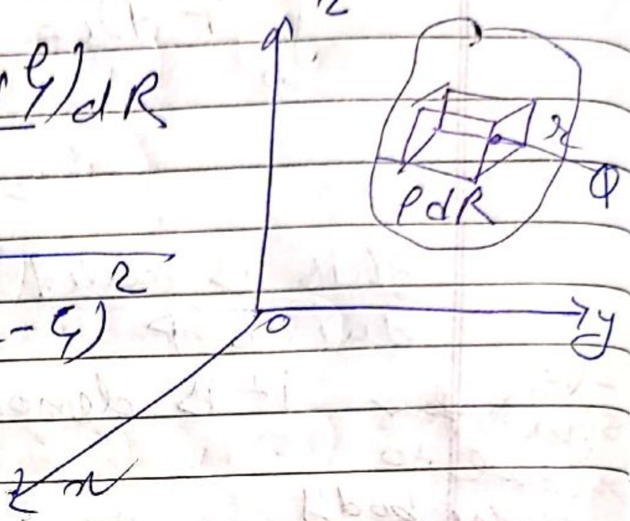
$$\int_{\infty}^{r} F\,dr = \sum_{K=1}^{n} \frac{m_K}{r_K} = -V$$

Now

$$\nabla^2 V = -\nabla^2 \sum_{K=1}^{n} \frac{m_K}{r_K} = -\sum_{K=1}^{n} \nabla^2\left(\frac{m_K}{r_K}\right)$$

$$= 0 \; ; \quad r_K \neq 0$$

IIIy for continuous distribution of mass in some volume R, we have



$$V(x,y,z) = \iiint_R \frac{\rho(\xi,\eta,\zeta)}{r}\, dR$$

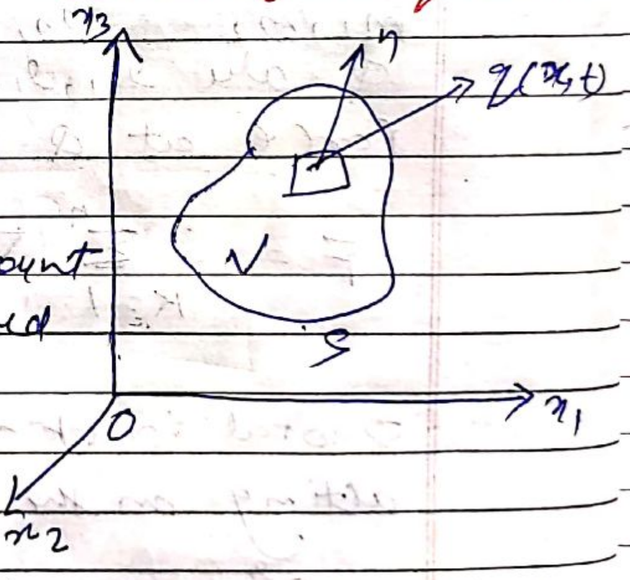
where

$$r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}$$

$$\Rightarrow \boxed{\nabla^2 V = 0.}$$

This equation is called Laplace equation also known as potential equation.

Conservation Laws and The Burgers Equation



A conservation law state that the rate of change of the total ~~volume~~ amount of material contained in a fixed domain of the volume V is equal to the flux of that material across the closed bounding surface S of the domain.

Let the density of material is $\rho(x,t)$ and flux vector is $q(x,t)$

(-) sign is because we took outward flux.

$\Rightarrow$ by conservation law

$$\frac{d}{dt}\int_V \rho\, dV = -\int_S (q\cdot n)\, dS. \quad \text{--- (1)}$$

where dV is the volume element & dS is the surface element and n is outward normal vector

using Gauss-divergence theorem

$$\int_V \left(\frac{\partial \rho}{\partial t} + \text{div}\, q\right) dV = 0 \quad \text{--- (2)}$$

if the integrand is continuous

$$\Rightarrow \boxed{\rho_t + \text{div}\, q = 0} \quad \text{--- (3)}$$

this is called differential form of the conservation law.

Now the one dimensional conservation law is

$$(4)\quad \rho_t + q_x = 0 \text{ i.e. } \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0}$$

for discontinuous solution or shock wave.

Assume $q = Q(\rho)$

or we can assume q is function of ρ & ρ_x both then

$$q = Q(\rho) - \nu \rho_x$$

where ν is ~~+ve~~ positive constant

(14)

Put the value of q in (4), we get

$$\rho_t + c(\rho)\,\rho_x = \nu\,\rho_{xx} \quad \text{--- (5)}$$

$$\left\{ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(Q(\rho) + \nu \rho_x\right) = 0 \right.$$
and apply chain rule $\}$

where $c(\rho) = Q'(\rho)$

Multiply by $c'(\rho)$ in (5)

$$c'(\rho)\,\rho_t + c'(\rho)\,c(\rho)\,\rho_x = \nu\, c'(\rho)\,\rho_{xx}$$

$$\Rightarrow \cancel{c_t}\; c_t + c\,c_x = \nu\left(c_{xx} - c''(\rho)\,\rho_x^2\right)$$

if $Q(\rho)$ is quadratic in ρ then
$c''(\rho) = 0$

$$\Rightarrow \quad c_t + c\,c_x = \nu\, c_{xx}$$

replace $u(x,t)$ by c
we get

$$\boxed{u_t + u\,u_x = \nu\, u_{xx}}$$ burger eqn.

where,
ν is the kinematic viscosity.