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$x(1-x)y'' - (1+3x)y' + y = 0$ (Same coeff ∞)

Sol^n — Given diff eq^n $x(1-x)y'' - (1+3x)y' + y = 0$ ——— (A)

Dividing by $x(1-x)$, we get

$$y'' - \frac{(1+3x)}{x(1-3x)}\,y' + \frac{1}{x(1-x)}\,y = 0 \quad \text{--- (1)}$$

Comparing (1) with $y'' + Py' + Qy = 0$ --- (2)

we get $P(x) = -\frac{(1+3x)}{x(1-x)} \Rightarrow (x-0)P(x) = -\frac{(1+3x)}{1-x} = -1 \neq \infty$ at $x=0$

& $Q(x) = \frac{1}{x(1-x)} \Rightarrow (x-0)^2 Q(x) = \frac{x}{1-x} = 0 \neq \infty$ at $x=0$

so both $xP(x)$ & $(x-0)^2Q(x)$ are analytic at $x=0$. So $x=0$ is regular singular point.

Now let the sol^n be of the form

$$y = x^m[a_0 + a_1x + a_2x^2 + \cdots] = \sum a_k x^{m+k} \quad \text{--- (3)}$$

$$y' = \sum a_k(k+m)x^{k+m-1} \quad \text{--- (4)}$$

$$y'' = \sum a_k(k+m)(k+m-1)x^{k+m-2} \quad \text{--- (5)}$$

putting (3), (4) & (5) into (A) we get

$$x(1-x)\sum a_k(k+m)(k+m-1)x^{k+m-2} - (1+3x)\sum a_k(k+m)x^{k+m-1} + \sum a_k x^{k+m} = 0$$

$$\Rightarrow \sum a_k(k+m)(k+m-1)x^{k+m-1} - \sum a_k(k+m)(k+m-1)x^{k+m} - \sum a_k(k+m)x^{k+m-1}$$
$$- 3\sum a_k(k+m)x^{k+m} + \sum a_k x^{k+m} = 0$$

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$$\Rightarrow \sum [a_k(k+m)(k+m-1) - a_k(k+m)]x^{k+m-1} - \sum [a_k(k+m)(k+m-1) + 3(k+m) + a_k]x^{k+m} = 0$$

$$\Rightarrow \sum a_k(m+k)(m+k-2)x^{k+m-1} - \sum a_k(m+k+1)^2 x^{k+m} = 0 \quad \text{--- (6)}$$

eqn is an identity.

Put $k=0$ we can get the coeff. of lowest degree term x^{m-1} in the first summation & equating it to zero. The indicial eqn is

$$a_0(m)(m-2) = 0$$

$$m(m-2) = 0$$

$$\Rightarrow \boxed{m=0, \quad m=2}$$

put equating to zero the coeff. of x^{k+m} in (6) we get

$$a_{k+1}(m+k+1)(m+k-1) - a_k(m+k+1)^2 = 0$$

$$\boxed{a_{k+1} = a_k \cdot \frac{(m+k+1)}{(m+k-1)}}$$ This is recurrence relation.

if $k=0$: $a_1 = a_0 \cdot \left(\frac{m+1}{m-1}\right)$

$k=1$: $a_2 = a_1\left(\frac{m+2}{m}\right) = a_0\left(\frac{m+1}{m-1}\right)\cdot\left(\frac{m+2}{m}\right)$

$k=2$: $a_3 = a_2\left(\frac{m+3}{m+1}\right) = a_0\left(\frac{\cancel{m+1}}{m-1}\right)\cdot\left(\frac{m+2}{m}\right)\cdot\frac{m+3}{\cancel{(m+1)}}$

$k=3$: $a_4 = a_3\left(\frac{m+4}{m+2}\right) = a_0 \cdot \frac{\cancel{(m+2)}}{(m-1)}\frac{(m+3)}{(m)}\cdot\frac{m+4}{\cancel{(m+2)}}$

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$m = 0$	$m = 2$
$a_1 = -a_0$	$a_1 = 3a_0$
$a_2 = \infty$	$a_2 = 6a_0$
$a_3 = \infty$	$a_3 = 6a_0 \cdot \frac{5}{3} = 10a_0$
$a_4 = \infty$	$\vdots$
$\vdots$	

but we are looking that ~~that~~ the if $m = 0$ some coefficient becomes infinite (∞).

So to remove this difficulty, we put

$$a_0 = (m-0)b \quad \text{or} \quad b(m-m_1)$$

$$\therefore a_0 = bm \quad ; \quad b \neq 0$$

$$\therefore y = x^m [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots]$$

$$= b x^m \left[m + \left(\frac{m+1}{m-1}\right) x + m\left(\frac{m+1}{m-1}\right)\left(\frac{m+2}{m}\right) x^2 + \cdots + m\left(\frac{m+3}{m-1}\right)\left(\frac{m+2}{m}\right) x^3 + \cdots \right]$$

$$= b x^m \left[m + \left(\frac{m+1}{m-1}\right) m \cdot x + \left(\frac{m+1}{m-1}\right)(m+2) x^2 + \left(\frac{m+3}{m-1}\right) \cdot (m+2) x^3 + \cdots \right]$$

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eqn (7) gives one solⁿ instead of two solutions.

The second solⁿ is given by

$$\left(\frac{\partial y}{\partial m}\right) = bx^m \log x \left[m + \frac{m(m+1)}{m-1}x + \frac{(m+1)(m+2)}{m-1}x^2 + \cdots\right]$$

$$+ bx^m \left[1 + \frac{(m-1)^2 (2m+1) + m(m+1)\,2(m-1)}{(m-1)^2} \cdot x + \frac{m^2-m-5}{(m-1)^2}x^2 + \frac{m^2-2m-1}{(m-1)^2}x^3 + \cdots\right]$$

$\therefore y =$

$$y = \left(\frac{\partial y}{\partial m}\right)_{m=0} = b\log x\left[\frac{2}{-1}x^2 + \cdots\right] + b\left[1 + \frac{(-1)}{1}x + (-5)x^2 + (-11)x^3 + \cdots\right]$$

$$= -b\log x\left[2x^2 + \cdots\right] + b\left[1 - x - 5x^2 - 11x^3 + \cdots\right]$$

$$= bv \quad \text{when } v = \text{…}$$

~~Hence required solⁿ can be obtained as~~

for $\underline{m=2}$

$$y_2 = a_0 x^2 \left[1 + 3x + 6x^2 + 10x^3 + \cdots\right] = a_0 u$$

$$\therefore y = c_2 y_2 + c_1 \left(\frac{\partial y}{\partial m}\right)_{m=m_1}$$

$$y = c_2 a_0 u + c_1 b v$$

$$\boxed{y = k_2 u + k_1 v}$$

Ans. $k_1 = c_1 b = \text{const.}$ & $k_2 = c_2 a_0 = \text{constt.}$

Legendre Differential Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

or $$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

Legendre Polynomial or Legendre function of first Kind $P_n(x)$

$$P_n(x) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2\cdot 2(n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right]$$

Legendre Polynomial or Legendre function of Second Kind $Q_n(x)$

$$Q_n(x) = \frac{n!}{1\cdot 3\cdot 5\cdots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}\cdot x^{-n-5} + \cdots\right]$$

Legendre eqn. may be expressed

$$\boxed{y = A\,P_n(x) + B\,Q_n(x)}$$

Where A & B are arbitrary constant

Generating Function of Legendre Polynomial

$$\boxed{(1-2zx+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)}$$

* Show that $P_n(1) = 1$

The generating function of legendre polynomial is

$$(1-2zx+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{-----} \quad (1)$$

sub. $x=1$ in (1)

$$(1-2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1)$$

$$\left[(1-z)^2\right]^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1)$$

$$(1-z)^{-1} = 1 + zP_1(1) + z^2 P_2(1) + \dots + z^n P_n(1)$$

$$1 + z + z^2 + \dots + z^n = 1 + zP_1(1) + z^2 P_2(1) + \dots + z^n P_n(1)$$

equating the coefficient of z^n on either side we get

$$\boxed{P_n(1) = 1}$$

Show that $P_n(-x) = (-1)^n P_n(x)$

Sol. The generating function of legendre polynomial is

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{------} (1)$$

St $x = -x$ in (1). we get

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-x) \quad \text{------} (2)$$

St. $z = -z$ in (1), we get

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n P_n(x)$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n P_n(x) \quad \text{------} (3)$$

comparing (2) & (3) we get

$$\sum_{n=0}^{\infty} z^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n z^n P_n(x)$$

$$\therefore \boxed{P_n(-x) = (-1)^n P_n(x)}$$

*** Show that $P_n(-1) = (-1)^n$ [PHY (H) 2001]

Sol: The generating function of Legendre Polynomial is

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{------} (1)$$

st. $x = -x$ in (1) we get

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-x) \quad \text{------} (2)$$

st. $z = -z$ in (1) we get

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n P_n(x)$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n P_n(x) \quad \text{------} (3)$$

Comparing (2) & (3) we get

$$\sum_{n=0}^{\infty} z^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n z^n P_n(x)$$

or $P_n(-x) = (-1)^n P_n(x)$ ------ (4)

st. $x = 1$ in (4) we get

$$P_n(-1) = (-1)^n P_n(1)$$

$$\boxed{P_n(-1) = (-1)^n} \quad [\text{Since } P_n(1) = 1]$$

** Rodrigues Formula for Legendre Polynomial $\boxed{2000}$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Sol: Let $y = (x^2-1)^n$ ---------- (1)

$$\frac{dy}{dx} = n(x^2-1)^{n-1}\cdot 2x$$

$$\frac{dy}{dx} = n(x^2-1)^n \cdot \frac{1}{(x^2-1)}\cdot 2x$$

or $(x^2-1)\frac{dy}{dx} = 2nx(x^2-1)^n$

or $(x^2-1)\frac{dy}{dx} = 2nxy$ (using (1)) ---------- (2)

Differentiating (2) $(n+1)$ times by Leibnitz Formula

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + {}^{n+1}C_1\frac{d^{n+1}y}{dx^{n+1}}\cdot(2x) + {}^{n+1}C_2\frac{d^ny}{dx^n}\cdot(2)$$
$$= 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}C_1\frac{d^ny}{dx^n}\cdot 1\right]$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2(n+1)x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^ny}{dx^n} = 2xn\frac{d^{n+1}y}{dx^{n+1}} + 2n(n+1)\frac{d^ny}{dx^n}$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x(n+1-n)\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\frac{d^ny}{dx^n} = 0$$

or [$\because$ LDE is $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$]

$$\Rightarrow (1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^ny}{dx^n} = 0 \quad \text{------ (3)}$$

Note :- $\boxed{P(n,r) = \frac{n!}{(n-r)!}}$ $\boxed{{}^nC_r = \frac{n!}{(n-r)!\,r!}}$

$$n! = n(n-1)!$$

Leibnitz Formula $\rightarrow$ $D^n(uv) = (D^n u)v + {}^nC_1(D^{n-1}u)Dv + {}^nC_2(D^{n-2}u)D^2v + \cdots\cdots + {}^nC_n(D^nv)u$

$D^n \rightarrow$ means differentiation n times.

st. $\frac{d^n y}{dx^n} = \phi(x)$ in (3), we get

$$(1-x^2)\frac{d^2\phi}{dx^2} - 2x\frac{d\phi}{dx} + n(n+1)\phi = 0 \quad \text{------} \quad (4)$$

This is Legendre differential eqn. has sol. $\phi = \frac{d^n y}{dx^n}$

Hence we may relate $\phi(x)$ with $P_n(x)$ as

$$P_n(x) = c\,\phi(x) \quad \text{---------} \quad (5)$$

where c is constt.

Now $y = (x^2-1)^n = (x-1)^n (x+1)^n$

diff. n times by Leibnitz Formula

$$\frac{d^n y}{dx^n} = (x-1)^n \frac{d^n}{dx^n}(x+1)^n + {}^nC_1\, n(x-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x+1)^n + \cdots$$

$$+ \cdots {}^nC_n (x+1)^n \boxed{\frac{d^n}{dx^n}(x-1)^n} \Rightarrow n!$$

put $x = 1$ on b.s

$$\left(\frac{d^n y}{dx^n}\right)_{x=1} = 2^n n! \qquad \left[\text{we know } \frac{d^n (x-1)^n}{dx^n} = n!\right]$$

st. $x = 1$ in (5)

$$P_n(1) = c\,\phi(x)\Big]_{x=1} = c\left(\frac{d^n y}{dx^n}\right)_{x=1} = c\,2^n n!$$

$$\therefore c = \frac{P_n(1)}{2^n n!} = \frac{1}{2^n n!} \qquad [\text{since } P_n(1) = 1]$$

$\therefore$ from (5)

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n y}{dx^n} = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n}$$

This Rodrigues formula.

Note $P_1(x) = \frac{1}{2^n n!}\frac{d^n y}{dx^n}$.

Prob. Sol. $\frac{d}{dx}(x-1)^1 = 1!$

$\frac{d^2}{dx^2}(x-1)^2 = \frac{d}{dx}\left[\frac{d}{dx}(x-1)^2\right] = \frac{d}{dx}(2(x-1))$

$= 2!$

$\frac{d^n}{dx^n}(x-1)^n = n!$

(3A)

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n} \text{ ---- Rodrigues formula}$$

$$\therefore\ P_0(x) = \frac{1}{2^0\, 0!}\frac{d^0}{dx^0}(x^2-1)^0 = 1$$

$$P_1(x) = \frac{1}{2\cdot 1!}\frac{d}{dx}(x^2-1) = \frac{1}{2}\cdot 2x = x$$

$$P_2(x) = \frac{1}{2^2\, 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{4\times 2}\frac{d}{dx}\left[\frac{d}{dx}(x^2-1)^2\right]$$

$$= \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)\cdot(2x)\right]$$

$$= \frac{1}{8}\times 4\frac{d}{dx}\left[x^3-x\right]$$

$$= \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3\, 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{8\times 3\times 2}\frac{d^2}{dx^2}\left[\frac{d}{dx}(x^2-1)^3\right]$$

$$= \frac{1}{8\times 3\times 2}\frac{d^2}{dx^2}\left[3(x^2-1)^2\cdot(2x)\right]$$

$$= \frac{\cancel{\times 3\times 2}}{8\cancel{\times 3\times 2}}\frac{d^2}{dx^2}\left[(x^4+1-2x^2)\,x\right]$$

$$= \frac{1}{8}\frac{d^2}{dx^2}\left[x^5-2x^3+x\right]$$

$$= \frac{1}{8}\frac{d}{dx}\left[5x^4-6x^2+1\right]$$

$$= \frac{1}{8}\left[20x^3-12x\right] = \frac{4}{8}\left[5x^3-3x\right]$$

$$= \frac{1}{2}\left[5x^3-3x\right]$$

(3B)

Similarly $P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]$

$$P_5(x) = \frac{1}{8}\left[63x^5 - 70x^3 + 15x\right]$$

$$P_6(x) = \frac{1}{16}\left[231x^6 - 315x^4 + 105x^2 - 5\right]$$

⋮

so on.

***** Orthogonal Properties of Legendre's Polynomials.**

(I) Show that $\int_{-1}^{+1} P_m(x)\, P_n(x)\, dx = 0$ for $m \neq n$

(II) Show that $\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1}$

Sol. (I) Legendre differential eqn is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

or $$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0 \quad \text{------ (1)}$$

as $P_m(x)$ & $P_n(x)$ are the sol. of Legendre ~~polynomial~~ eqn (1)

$$\therefore \frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] + n(n+1)P_n = 0 \quad \text{------ (2)}$$

$$\& \quad \frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right] + m(m+1)P_m = 0 \quad \text{------ (3)}$$

multiply (2) by P_m & (3) by P_n & subtracting we get

$$P_m \frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right] - P_n \frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right] + P_n P_m [n(n+1) - m(m+1)] = 0$$

Integrating within limits -1 to $+1$ w.r.t x we get

$$\int_{-1}^{+1} \underset{\mathrm{I}}{P_m} \underbrace{\frac{d}{dx}\left[(1-x^2)\frac{dP_n}{dx}\right]}_{\mathrm{II}} dx - \int_{-1}^{+1} \underset{\mathrm{I}}{P_n} \underset{\mathrm{II}}{\frac{d}{dx}\left[(1-x^2)\frac{dP_m}{dx}\right]} dx$$
$$+ (n-m)(n+m+1)\int_{-1}^{+1} P_n P_m\, dx = 0$$

Integrate by parts

* Trick: in odd we want x terms, in even free from x.

$$\underbrace{\left[P_m (1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1}}_{=0} - \int_{-1}^{+1}\frac{dP_m}{dx}\left[\cancel{(1-x^2)}\frac{dP_n}{dx}\right]dx$$

$$-\underbrace{\left[P_n(1-x^2)\frac{dP_m}{dx}\right]_{-1}^{+1}}_{=0} + \int_{-1}^{+1}\frac{dP_n}{dx}\left[\cancel{(1-x^2)}\frac{dP_m}{dx}\right]dx + (n-m)(n+m+1)\int_{-1}^{+1}P_n P_m\,dx = 0$$

or $$0 - \int_{-1}^{+1}\frac{dP_m}{dx}\left[\cancel{(1-x^2)}\frac{dP_n}{dx}\right]dx - 0 + \int_{-1}^{+1}\left[\frac{dP_n}{dx}\cancel{(1-x^2)}\frac{dP_m}{dx}\right]dx + (n-m)(n+m+1)\int_{-1}^{+1}P_n P_m\,dx = 0$$

or $$(n-m)(n+m+1)\int_{-1}^{+1}P_n P_m\,dx = 0$$

$$\boxed{\int_{-1}^{+1}P_n P_m\,dx = 0} \quad \text{if } m \neq n$$

(II) Sol: Generating function is $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$

squaring both sides & integrating within limit -1 to $+1$

$$\int_{-1}^{+1}(1-2xz+z^2)^{-1}dx = \sum_{n=0}^{\infty}\int_{-1}^{+1} z^{2n}[P_n(x)]^2 dx$$

or $$-\frac{1}{2z}\left[\log(1-2xz+z^2)\right]_{-1}^{+1} = \sum_{n=0}^{\infty} z^{2n}\int_{-1}^{+1}[P_n(x)]^2 dx$$

or $$-\frac{1}{2z}\left[\log(1-z)^2 - \log(1+z)^2\right] = \sum_{n=0}^{\infty} z^{2n}\int_{-1}^{+1}[P_n(x)]^2 dx$$

or $$\frac{1}{z}\left[\log(1+z) - \log(1-z)\right] = \sum_{n=0}^{\infty} z^{2n}\int_{-1}^{+1}[P_n(x)]^2 dx$$

or $\frac{1}{2}\left[\left(z-\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right)-\left(-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots\right)\right]=\sum_{n=0}^{\infty} z^{2n}\int_{-1}^{+1}[P_n(x)]^2\,dx$

or $\frac{2}{2}\left[z+\frac{z^3}{3}+\frac{z^5}{5}+\cdots+\frac{z^{2n+1}}{2n+1}\right]=\sum_{n=0}^{\infty} z^{2n}\int_{-1}^{+1}[P_n(x)]^2\,dx$

or $2\left[1+\frac{z^2}{3}+\frac{z^4}{5}+\cdots+\frac{z^{2n}}{2n+1}\cdots\right]=\sum_{n=0}^{\infty} z^{2n}\int_{-1}^{+1}[P_n(x)]^2\,dx$

Equating the coefficient of z^{2n} on both sides we get

$$\boxed{\int_{-1}^{+1}[P_n(x)]^2\,dx=\frac{2}{2n+1}}$$