(2) The set $\mathbb{R}^2 = \{(a_1, a_2) \mid a_1, a_2 \in R\}$ is a group under composition addition

(Soln) let $x = (a_1, a_2) \in \mathbb{R}^2$ s.t $a_1, a_2, b_1, b_2 \in R$
$y = (b_1, b_2) \in \mathbb{R}^2$

then $x + y = (a_1, a_2) + (b_1, b_2)$
$= (a_1 + b_1, a_2 + b_2)$
$\in R^2$ ($\because a_1 + b_1 \in R$ & $a_2 + b_2 \in \mathbb{R}$)

(i) Associative law!
let $x, y, z = (c_1, c_2) \in R^2$

we set
$(x + y) + z = x + (y + z)$ $\forall$ $x, y, z \in \mathbb{R}^2$

(iii) $\exists$ $0 = (0, 0) \in \mathbb{R}^2$ s.t
$x + 0 = (a_1, a_2) + (0, 0) = (a_1, a_2) = x$
and $0 + x = (0, 0) + (a_1, a_2) = (0, a_2) = x$
$\forall$ $x \in \mathbb{R}^2$

(iv) for each $x = (a_1, a_2) \in \mathbb{R}^2$
$\exists$ $x' = (-a_1, -a_2) \in \mathbb{R}^2$ s.t
$x + x' = (a_1 - a_1, a_2 - a_2) = (0, 0) = 0$
$x' + x = 0$
$\therefore (\mathbb{R}^2, +)$ is group

Group $U(n)$, $n>1$ wrt (or under) multiplication modulo n

$U(n) = \{ x \in \mathbb{N} \mid 1 \le x < n \text{ and relatively prime to } n \}$

$= \{ x \in \mathbb{N} \mid 1 \le x < n \text{ and } \gcd(x, n) = 1 \}$

$U(n)$ is set of all positive integers less than n and relatively prime to n.

Consider for $n=10$, we have $U(10) = \{1, 3, 9, 7\}$

$|U(10)| = \phi(10) = \phi(2 \times 5)$

$= \phi(2) \times \phi(5)$

$= 1 \times 4 = 4$

The Cayley table for $U(10)$ is

mod 10	1	3	7	9
1	(1)	3	7	9
3	3	9	(1)	7
7	7	(1)	9	3
9	9	7	3	(1)

① $\forall\, a \odot_{10} b \in U(10)$ wrt to multiplication modulo 10

② Associative true

③ 1 is identity

$\forall a \in U(10)\ \exists\ 1 \in U(10)$ s.t $a \odot_{10} 1 = 1 \odot_{10} a = a$

④ For each $a \in U(10)$ $\exists\ b \in U(10)$ s.t $a \odot b = b \odot a = 1$

Inverse of 7 is 3

of 9 is 9

of 1 is 1

of 3 is 7

$\therefore U(10)$ is group under multiplication modulo 10.

(a) Show that $U(n)$; $n>1$ is group w.r.t multiplication

(Soln) $U(n) = \{x \in \mathbb{N} \mid 1 \le x < n \ \& \ gcd(x, n) = 1\}$

and operation is multiplication modulo n.

(I) Let $a, b \in U(n)$ then $1 \le a < n$ & $gcd(a, n) = 1$

$1 \le b < n$ & $gcd(b, n) = 1$

As $gcd(a, n) = 1$

& $gcd(b, n) = 1 \Rightarrow gcd(ab, n) = 1$

then $a \cdot b \in U(n)$ under multiplication modulo n

(II) $(a \cdot b) c = a \cdot (b \cdot c)$ $\forall$ $a, b, c \in U(n)$

(III) Since $n > 1$ then $1 \in U(n)$ s.t

$a \cdot 1 = 1 \cdot a = a$ $\forall$ $a \in U(n)$ (since $gcd(1, 1) = 1$)

(IV) For each $a \in U(n)$, $\exists$ $a^{-1} \in U(n)$ s.t

$a \cdot a^{-1} = a^{-1} \cdot a = 1$

then $U(n)$ is group w.r.t to multiplication modulo n.

In particular if n is prime, then $U(n) = \{1, 2, \ldots, n-1\}$

(positive integers)

(e.g) $U(5) = \{1, 2, 3, 4\}$ is abelian group under $\odot_5$ multiplication modulo 5.

$\odot_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

By composition table

(1) closure holds

(2) Associative

(3) 1 identity

(4) inverse of 1 is 1

2 is 3

3 is 2

4 is 4

It is also abelian group.

$(Z_n, \oplus)$; $n > 1$ is always abelian

$U(n)$; $n > 1$ is always abelian.

Remarks:

① $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbb{R},\ 1 \le i \le n\}$

$(\mathbb{R}^n, +)$ is group.

② General linear group, $GL(n, \mathbb{R})$

$GL(n, \mathbb{R})$ = { set of all $n \times n$ invertible matrix over $\mathbb{R}$ }

i.e $n \times n$ (matrix) $= A = \{[a_{ij}]_{n \times n} \mid |A| \neq 0 \ \& \ a_{ij} \in \mathbb{R}\}$

$GL(2, \mathbb{R})$ is group under matrix addition

③ Special linear group, $SL(n, \mathbb{R})$

$SL(n, \mathbb{R})$ = {set of $n \times n$ matrix with $|A| = 1$ over $\mathbb{R}$ }

$= \{A = [a_{ij}]_{n \times n} \mid |A| = 1\}$

$a_{ij} \in \mathbb{R}$

$SL(2, \mathbb{R})$ is group under matrix multiplication.

④ If be any of $\mathbb{Q}, \mathbb{R}$, & $\mathbb{Z}_p$ (p is prime)

$GL(2, F)$ with non-zero determinant & entries from F

is Non-abelian group under matrix multiplication.