(52) Some Consequences of the extension

Theorem (1) If a function f is analytic at a given point then its derivative of all orders are analytic there too.

(proof) let function 'f' is analytic at a point z_0. then there must be a nbd $|z-z_0| < \epsilon$ of z_0 throughout which f is analytic

Consequently, there is positively oriented circle C_0, centered at z_0 with radius $\epsilon/2$ s.t f is analytic inside and on C_0.

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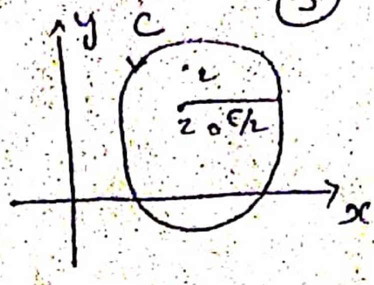

As f is analytic inside & on C_0

$\therefore$ we know that

$$f''(z) = \frac{1}{\pi i}\int_{C_0} \frac{f(s)\,ds}{(s-z)^3}$$

at each point z interior to C_0

and existence of $f''(z)$ throughout the nbd $|z-z_0| < \epsilon/2$ implies

f' is analytic at z_0.

i.e By analytic function f', we get derivative f'' is analytic

similarly, we get ~~others~~ required result.

Corollary: If a function $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z=(x,y)$, then the component functions u and v have continuous partial derivatives of all orders at that point.

(Pf) let $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z=(x,y)$

then differentiability of f' ensures the continuity of f', then

$$f'(z) = u_x + iv_x$$
$$= v_y - iu_y \quad (\text{by C.R eq}^{ns})$$

Also, first-order partial derivatives of u & v are cts at that point.

By sufficient cond^n of analytic of a fc^n f is analytic we get f'' is analytic & cts at z (by above thm)

and $$f''(z) = u_{xx} + iv_{xx}$$
$$= v_{yx} - iu_{yx}$$

similarly, we get required result.

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Theorem (2) Let f be cts on domain D. If
$$\int_C f(z)\,dz = 0$$
for every closed contour in D
then f is analytic throughout in D.

(Pf) Let f be cts on D & $\int_C f(z)\,dz = 0$ for every closed contour in D

By Antiderivative equivalence cases

f has Antiderivative in D

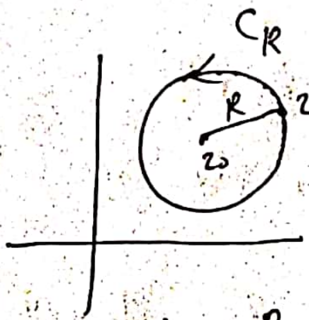
i.e there exists an analytic fun^n F s.t
$$F'(z) = f(z) \text{ at each point in } D$$

Since f is the derivative of F

by theorem (2) F is analytic $\Rightarrow F'$ is analytic
$\Rightarrow f$ is analytic

Theorem (3) Suppose that a function f is analytic inside & on a positively oriented circle C_R centered at z_0 with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R then

$$|f^n(z_0)| \leq \frac{n!\,M_R}{R^n} \quad (n=1,2 \ldots)$$



(Pf) By using extension of Cauchy's inequality

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad (n=1,2 \ldots)$$

Consider

$$|f^n(z_0)| = \left|\frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z-z_0)^{n+1}}\right|$$

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$$= \frac{n!}{2\pi i}\int_{C_R}^{2\pi} \frac{f(z_0+Re^{i\theta})}{(Re^{i\theta})^{n+1}}\, Re^{i\theta}\, i\, d\theta \qquad \begin{array}{l} z = z_0 + Re^{i\theta} \\ dz = iRe^{i\theta}\, d\theta \end{array}$$

$$= \left|\frac{n!}{2\pi i}\int_0^{2\pi} \frac{f(z_0+Re^{i\theta})}{(Re^{i\theta})^{n+1}}\, Re^{i\theta}\, i\, d\theta\right|$$

$$\le \frac{n!}{2\pi}\int_0^{2\pi} \frac{|f(z_0+Re^{i\theta})|}{|(Re^{i\theta})^{n+1}|}\, |Re^{i\theta}|\, |i|\, d\theta$$

$(|i|=1)$

$$= \frac{n!}{2\pi} M \int_0^{2\pi} \frac{|Re^{i\theta}|\, d\theta}{|(Re^{i\theta})^{n+1}|}$$

$$= \frac{n!}{2\pi} M \frac{R}{R^{n+1}} \int_0^{2\pi} d\theta$$

$$= \frac{n!\, M_R}{2\pi} \frac{1}{R^n}\, 2\pi$$

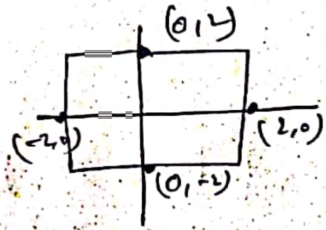
$$= \frac{n!\, M_R}{R}$$

$$\therefore\ |f^n(z_0)| \le \frac{n!\, M_R}{R^n} \quad (n=1, \ldots)$$

(Exe) (Pg 170)

(Q-1) C is positively oriented boundary of square whose sides lie along the lines $x=\pm 2$ & $y=\pm 2$

(a) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\, dz$



$\frac{\cos z}{z(z^2+8)}$ is not analytic at $z=0$

$z = \pm 2\sqrt{2}$

$z = 0 + i0$

$z = \pm 2\sqrt{2} + i0$

$z=0$ lies only inside C & $z = |2\sqrt{2}|$

$= 8 > C$

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$$\int_C \frac{\cos z}{z(z^2+8)}\,dz = \int_C \frac{\frac{\cos z}{z^2+8}}{z-0}\,dz$$

By Cauchy's formula

$$= 2\pi i\, f\left.\frac{\cos z}{(z^2+8)}\right|_{z=0} \qquad f_0\; f(z) = \frac{\cos z}{z^2+8}$$

$$= 2\pi i \frac{1}{8}$$

$$= \frac{\pi i}{4}$$

(b) $\int_C \frac{\cosh z}{z^4}\,dz$

is not analytic at $z=0$

$z=0$ lies inside of C

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(z_0) \qquad \text{where } f(z) = \cosh z$$

$$\int_C \frac{\cosh z}{(z-0)^{3+1}} = \frac{2\pi i}{3}(\cosh z)^{'''}_{z=0}$$

$$= \frac{2\pi i}{3} \sinh z\Big|_{z=0}$$

$$= 0$$

(c) $\int_C \frac{\tan(z/2)}{(z-x_0)^2}$ where $-2 < x_0 < 2$

$z = x_0$ is point where it is not analytic (lies in C)

$$\int_C \frac{\tan z/2}{(z-x_0)^2} = \frac{2\pi i}{1} f'(x_0) \qquad f(z) = \tan z/2$$

$$n=1 \qquad = 2\pi i \tfrac{1}{2} \sec^2 z/2\Big|_{x_0}$$

$$= \pi i \sec^2 \frac{x_0}{2}$$

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case (1) $|S| < 3$ inside C

So, (formula) $f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(S)}{(S-z)^{n+1}} dS$

Now $g(z) = \int_C \frac{(2S^2 - S - 2)}{S-2} dS$ → analytic $\qquad g(S) = 2S^2 - S - 2$

$$g(z) = g^0(z) = \frac{0!}{2\pi i} \int_C \frac{2S^2 - S - 2}{(S-2)} dS$$

$$2\pi i \, (2S^2 - S - 2)_{z=2} = \int_C \frac{2S^2 - S - 2}{(S-2)} dS$$

$\Rightarrow 2\pi i (4)$

$\Rightarrow 8\pi i$

on other hand when $|z| > 3$

$|z| = |S| > 3$

so, outside C

Now $\frac{2S^2 - S - 2}{S-2}$ is analytic at all points of an simple closed contour

$\therefore$ By Cauchy-Goursat theorem

$$\int_{|z|>3} g(z) = 0$$

(Q-4) similar

(Q-5) Show that if f is analytic within & on simple closed contour C & z_0 is not on C

then $\int_C \frac{f'(z)\, dz}{z - z_0} = \int_C \frac{f(z)\, dz}{(z-z_0)^2}$

(Sol) let f is analytic within & on closed (simple) contour C & z_0 is not on C. If z_0 is inside C

then By Cauchy integral formula (extension)

$$\int_C \frac{f'(z)\, dz}{z - z_0} = 2\pi i \, f'(z_0)$$

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and $\int \frac{f(z)\,dz}{(z-z_0)^2} = \int_C \frac{f(z)\,dz}{(z-z_0)^{1+1}} = \frac{2\pi i}{1!} f'(z_0)$

$\therefore \int_C \frac{f'(z)\,dz}{z-z_0} = \int \frac{f(z)\,dz}{(z-z_0)^2}$ — (1) using $f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)\,ds}{(s-z)^{n+1}}$

If z_0 is exterior to C
then eqn (1) is also valid
by Cauchy Goursat theorem
both equal to zero.

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