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Q. 1, $\underset{\text{even} + \text{odd} = \text{odd}}{(5)(2)}$, $\underset{\text{odd}}{(10)}$ there is no element of order 10

$\underset{\text{even} + \text{even} = \text{even}}{(5)(3)}$ or $\underset{\text{even}}{(15)}$

$$\frac{(15)(14)(13)(12)(11)}{(5)} \times \frac{(10)(9)(8)}{3} + \frac{(15)(14)(13)\cdots(3)(2)(1)}{15} = \underline{\qquad\qquad}$$

$2\mathbb{Z}$ & $3\mathbb{Z} \triangleleft \mathbb{Z}$

Q. 4 No, $2\mathbb{Z} \cup 3\mathbb{Z} \ntriangleleft \mathbb{Z}$

5 $\phi : S_n \to \{-1, 1\}$

$$\phi(\alpha) = \begin{cases} 1, & \text{if } \alpha \text{ is even permutation} \\ -1, & \text{if } \alpha \text{ is odd permut} \end{cases}$$

$\ker \phi = A_n$ & $\phi$ is onto

First Isomorphism thm, $\frac{S_n}{A_n} \cong \{-1, 1\}$

Q. 6 Yes, Converse is true for cyclic groups

If $d|n$ & $|G|=n$, G must have subgroup of order d

Fundamental thm of cyclic group.

$\langle a^{n/d} \rangle$

Q. 7 No, $U(15)=\{1,2,4,7,8,11,13,14\}$

$U(24)=\{1,5,7,11,13,17,19,23\}$

no. of solⁿs of $x^k=1$ & no. of solⁿs of $x^k=\phi(1)$

$|A|=|B|$

$x^2=1 \rightarrow$ in $U(15)$ ....

one-one onto.

$\boxed{\text{O.P.}}$

$x^2=1 \rightarrow$ In $U(24)$ - - -

Q. 8 $|G|=77$

elements of order 1, 7, 11 or 77

If 77 order element is in G

$\Rightarrow$ G is cyclic

elements of order 11 $= \phi(11) = 10$

If element of order 77 is not in G.

If there are 30 elements of order 11

then there must be ~~4~~6 elements of order 7

But element of order 7 must be multiple of $\phi(7) = 6$

But 46 is not a multiple of 6

$\longrightarrow \Leftarrow$

$\therefore$ there can't be exactly 30 elements of order 11.

Q (9) yes,

$H = \{ \alpha \in A_6 \mid \alpha(2) = 4 \} \cong A_5$

$K = \{ \beta \in A_6 \mid \beta(2) = 5 \} \cong A_5$

$H \cong K \cong A_5$

Q. 10 No, $|G| = 40$, $|Z(G)| = 20$.

$$\left|G/Z\right| = 40/20 = 2 \quad (\text{prime})$$

$\Rightarrow G/Z(G)$ is cyclic $\Rightarrow G$ is abelian (G/Z thm)

$\Rightarrow G = Z(G)$

$\Rightarrow |G| = |Z(G)|$

$\longrightarrow \Leftarrow$

Q. 12 $\dfrac{R}{\langle 2\pi \rangle} \cong GL(2, R)$

$\phi : R \to GL(2, R)$ such that

$$\phi(x) = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$$

$\phi$ is homomorphism & onto

$\ker \phi = \langle 2\pi \rangle$

first Isomorphism thm, $\dfrac{R}{\langle 2\pi \rangle} \cong GL(2, R)$

Q. 11 All homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_6$

Sol// $\phi(x) = ax$ ~~for~~ $a = 0, 1, 2, 3, 4, 5$.

$\phi(1) = a \Rightarrow |a|$ divides $|1|$

$\Rightarrow |a|$ divides 12

$\Rightarrow |a| = 1, 2, 3, 4, 6$ or 12.

$\because \phi(1) = a \Rightarrow a \in \mathbb{Z}_6 \Rightarrow |a|$ divides 6

$\Rightarrow |a| = 1, 2, 3$ or 6

$\Rightarrow a = 0, 3, 2, 4, 1$ or 5.