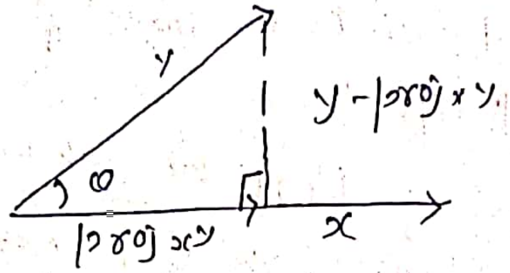
(7.4) Orthogonal Projection

As in chapter-1, we have result
If x is non-zero vector in $\mathbb{R}^n$, then every vector y in $\mathbb{R}^n$ can be decomposed as the sum of two component vectors, $\text{proj}_x y$ and $y - \text{proj}_x y$, where the first is parallel to x and second is orthogonal to x.



Theorem (7.12) Projection theorem
let W be subspace of $\mathbb{R}^n$. Then every vector v in $\mathbb{R}^n$ can be written uniquely in the form $v = w_1 + w_2$ — ①
where w_1 is in W & w_2 is in $W^{\perp}$.
In fact, if $\{u_1, u_2, \ldots, u_k\}$ is any orthonormal basis for W,
then $w_1 = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \cdots + (v \cdot u_k)u_k$ — ②
and $w_2 = v - w_1$
The vector w_1 in ①, is called orthogonal projection of v onto W and written as $\text{proj}_W v$.

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Definition: Orthogonal projection onto a Subspace

let W be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{u_1, u_2, ---, u_k\}$ and let $v \in \mathbb{R}^n$. Then orthogonal projection of v onto W is defined to the vector

$$\text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2 + \cdots + (v \cdot u_k) u_k$$

If W is trivial subspace of $\mathbb{R}^n$, then

$$\text{proj}_W v = 0$$

Another form of Projection theorem:

let W be subspace of $\mathbb{R}^n$. Then every vector v in $\mathbb{R}^n$ can be expressed in a unique way as $w_1 + w_2$ where

$$w_1 = \text{proj}_W v \in W$$

and $w_2 = v - \text{proj}_W v \in W^{\perp}$

Moreover, w_2 can be expressed as $\text{proj}_{W^{\perp}} v$.

Example (13) Find orthogonal projection of $v = [-1, 4, 3]$ onto subspace W of $\mathbb{R}^3$ spanned by orthogonal vectors $v_1 = [1, 1, 0]$ and $v_2 = [-1, 1, 0]$

Solution: The subspace W of $\mathbb{R}^3$ is defined to be $W = \text{span}(\{[1,1,0], [-1,1,0]\})$

$$= \text{span}\left\{\frac{1}{\sqrt{2}}[1,1,0], \frac{1}{\sqrt{2}}[-1,1,0]\right\}$$

where, normalized the vector $[1, 1, 0]$ is $\frac{1}{\sqrt{2}}[1,1,0]$ & vector $[-1,1,0]$ is $\frac{1}{\sqrt{2}}[-1,1,0]$

get orthonormal basis for W.

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Now,

$\text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2$ ②

where $u_1 = \frac{1}{\sqrt{2}}[1,1,0]$ & $u_2 = \frac{1}{\sqrt{2}}[-1,1,0]$

$\therefore \text{proj}_W v = \left(\underset{(v)}{[-1,4,3]} \cdot \underset{(u)}{\left(\frac{1}{\sqrt{2}}[1,1,0]\right)} \right) \underset{(u_1)}{\left(\frac{1}{\sqrt{2}}[1,1,0]\right)}$

$+ \left(\underset{(v)}{[-1,4,3]} \cdot \underset{(u_2)}{\frac{1}{\sqrt{2}}[-1,1,0]} \right) \underset{(u_2)}{\left(\frac{1}{\sqrt{2}}[-1,1,0]\right)}$

$= \frac{3}{2}[1,1,0] + \frac{5}{2}[-1,1,0]$

$= [-1,4,0]$

Example ⑭ Consider the subspace $W = \text{span}(\{[1,-2,-1], [3,-1,0]\})$ of $\mathbb{R}^3$. Let $v = [-1,3,2]$. Find $\text{proj}_W v$ and decompose v into $w_1 + w_2$ where $w_1 \in W$ & $w_2 \in W^\perp$.

Is the decomposition unique?

Solution Let $x_1 = [1,-2,-1]$ & $x_2 = [3,-1,0]$

Neither of these vectors is scalar multiple of other, we get x_1 & x_2 are L.I

$\therefore \{x_1, x_2\}$ form a basis for W

(since $W = \text{span}[\{x_1, x_2\}]$ & $\{x_1, x_2\}$ are L.I)

Use Gram-Schmidt Process on basis $\{x_1 = [1,-2,-1]$ & $x_2 = [3,-1,0]\}$ for W to obtain orthogonal basis for W.

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Following Gram-Schmidt Process, we have

$v_1 = x_1 = [1, -2, -1]$

$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{\|v_1\|^2}\right) v_1$

$= [3,-1,0] - \frac{[3,-1,0] \cdot [1,-2,-1]}{6} [1,-2,-1]$

$= [3,-1,0] - \frac{5}{6}[1,-2,-1] = \left[\frac{13}{6}, \frac{4}{6}, \frac{5}{6}\right]$

To avoid fraction, we multiply vector by 6.

we get $v_2 = [13, 4, 5]$

Also, v_2 orthogonal to v_1

Hence $\{v_1, v_2\} = \{[1,-2,-1], [13,4,5]\}$ is an orthogonal basis for W.

Now, Normalizing v_1 & v_2, we obtain

$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{6}}[1,-2,-1]$

$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{210}}[13,4,5]$

Hence, $\{u_1, u_2\}$ are orthonormal basis for W.

$\therefore w_1 = \text{proj}_W v = (v \cdot u_1)u_1 + (v \cdot u_2)u_2$

$= \left(\underset{(v)}{[-1,3,2]} \cdot \left(\underset{(u_1)}{\frac{1}{\sqrt{6}}[1,-2,-1]}\right)\right)\left(\underset{(u_1)}{\frac{1}{\sqrt{6}}[1,-2,-1]}\right)$

$+ \left(\underset{(v)}{[-1,3,2]} \cdot \left(\underset{(u_2)}{\frac{1}{\sqrt{210}}[13,4,5]}\right)\right)\left(\underset{(u_2)}{\frac{1}{\sqrt{210}}[13,4,5]}\right)$

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$= \frac{1}{6}(-7)[1,-2,-1] + \frac{1}{210}(7)[13,4,5]$

$= \left[-\frac{33}{35}, \frac{111}{35}, \frac{12}{7}\right]$

and $w_2 = v - \text{proj}_W v$

$= [-1, 3, 2] - \left[-\frac{33}{35}, \frac{111}{35}, \frac{12}{7}\right]$

$= \left[-\frac{2}{35}, -\frac{6}{35}, \frac{2}{7}\right]$

Since w_2 is orthogonal to both x_1 & x_2

$\therefore w_2 \in W^\perp$

Hence we get decomposed

$v = [-1, 3, 2]$

as sum of w_1 & w_2 where $w_1 \in W$

$w_2 \in W^\perp$.

Also, this decomposition is unique as $W \cap W^\perp = \{0\}$.

Distance from a Point to a subspace : An applicat

Definition Minimum Distance

Let W be a subspace of $\mathbb{R}^n$ and assume that all vectors in W have initial point at origin. Let P be any point in n-dimensional space. Then the minimum distance from P to W is defined as shortest distance between P & the terminal point of any vector in W.

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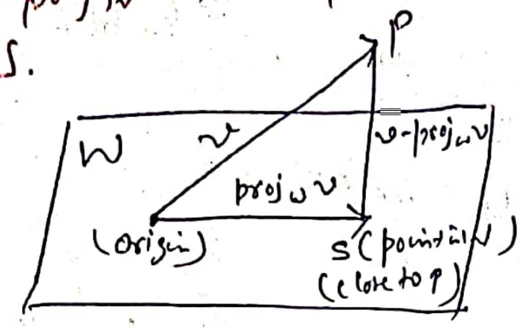
Theorem (7.13)
let W be subspace of $\mathbb{R}^n$, let P be point in n-dimensional space. If v is vector from the origin to P, then minimum distance from P to W is given by $\|v - proj_W v\|$.

Note: If S is terminal point of $proj_W v$ then $\|v - proj_W v\|$ represents distance from P to S.



Example (17) let W be subspace of $\mathbb{R}^3$ whose vector lie in plane $2x+y+z=0$. Find the minimum distance from the point $P(-6,10,5)$ to W.

Solution let v be vector from origin to point $P(-6,10,5)$.
Then minimum distance from $P(-6,10,5)$ to W is $\|v - proj_W v\|$

we have $v - proj_W v = \left[1, \frac{1}{2}, \frac{1}{2}\right]$

Hence minimum distance from $P(-6,10,5)$ to W is

$$\|v - proj_W v\| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{3}{2}}$$

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