Ch-2 Linear Transformations & Matrices

In Chapter 1, we studied the theory of vector spaces. In this chapter we are going to do linear transformations which are functions over vector spaces and preserve the structure of a vector space in some sense.

Section 2.1

Def^n:- Let V and W be vector spaces over same Field F. We call a function $T: V \to W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in F$, we have,

(a) $T(x+y) = T(x) + T(y)$ and

(b) $T(cx) = cT(x)$.

We often simply call T linear.

Properties:- Let $T: V \to W$ be a function where V and W are Vector spaces (over F).

1.) If T is linear, then $T(0) = 0$

Proof:- As T is linear

$\therefore$ $T(cx) = cT(x)$ for all $c \in F$ and $x \in V$.

$\Rightarrow$ $T(0) = T(0.x) = 0.T(x) = 0$ $[\because 0.x = 0 \;\forall x \in V]$.

$\Rightarrow$ $T(0) = 0$.

(2.) T is linear if and only if $T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$.

<u>Proof</u>:- Let $T(cx+y) = cT(x) + T(y) \quad \forall\, x, y \in V \,\&\, c \in F$.

<u>Claim</u>:- T is linear.

Taking $c = 1$, we get

$$T(x+y) = T(x) + T(y), \quad \forall\, x, y \in V$$

and taking $y = 0$, we get

$$T(cx) = cT(x), \quad \forall\, x \in V \,\&\, c \in F.$$

<u>Conversely</u>, let T is linear.

$\therefore$ (a) $T(x+y) = T(x) + T(y), \quad \forall\, x, y \in V$

(b) $T(cx) = cT(x), \quad \forall\, x \in V, \,\&\, c \in F.$

Let $c \in F$ and $x \in V$ & $y \in V$

then $cx \in V$

$\therefore$ By (a) $T(cx+y) = T(cx) + T(y)$

$= cT(x) + T(y)$ (By (b)

$\therefore \quad T(cx+y) = cT(x) + T(y), \quad \forall\, x, y \in V \,\&\, c \in F.$

(3.) If T is linear, then $T(x-y) = T(x) - T(y) \quad \forall\, x, y \in V$.

<u>Proof</u>:- Let $x, y \in V$, then $-y \in V$

$\therefore T(x + (-y)) = T(x) + T(-y)$

$= T(x) + T((-1)y)$ $\left[\begin{array}{l} \text{~~As (…)~~} \\ \text{As } (-1)x = -x \\ \quad = 1.(-x) \end{array}\right]$

$$\Rightarrow \quad = T(x) + (-1)\,T(y)$$

$$T(x+(-y)) = T(x) - T(y)$$

$$\therefore \; T(x-y) = T(x+(-y)) = T(x) - T(y)$$

$$\Rightarrow \; T(x-y) = T(x) - T(y), \; \forall\, x, y \in V.$$

(4.) T is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i).$$

Proof:- Let T is linear, we will prove the required claim by mathematical induction on n.

If $n=1$, then $T(a_1 x_1) = a_1 T(x_1)$ $\forall\, x_1 \in V$ & $a_1 \in F$

As T is linear ($\because$ By (b) part)

$\Rightarrow$ The result is true for $n=1$.

Now the result is true for $n=k-1$, i.e.

$$T\left(\sum_{i=1}^{k-1} a_i x_i\right) = \sum_{i=1}^{k-1} a_i T(x_i) \qquad -(1)$$

for all $x_1, x_2, \ldots, x_{k-1} \in V$ and $a_1, a_2, \ldots, a_{k-1} \in F$.

Now we show that the result is true for $n=k$.

Consider, $x_1, x_2, -, x_k \in V$ and $a_1, a_2, -, a_k \in F$, then

$$T\left(\sum_{i=1}^{k} a_i x_i\right) = T\left(\sum_{i=1}^{k-1} a_i x_i + a_k x_k\right)$$

$$= T\left(\sum_{i=1}^{k-1} a_i x_i\right) + T(a_k x_k) \quad \left[\text{By } \cancel{\text{part}}\ (a) \text{ of part}\right]$$

$$= T\left(\sum_{i=1}^{k-1} a_i x_i\right) + a_k T(x_k) \quad \left[\text{By } (b) \text{ part}\right]$$

$$= \sum_{i=1}^{k-1} a_i T(x_i) + a_k T(x_k) \quad \left[\text{By eq. } (1)\right]$$

$$T\left(\sum_{i=1}^{k} a_i x_i\right) = \sum_{i=1}^{k} a_i T(x_i)$$

which is required.

$\therefore$ By Principle of Mathematical induction, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

for $x_1, x_2, --, x_n \in V$ and $a_1, a_2, -, a_n \in F$.

__Conversely__ let $T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$

for $x_1, x_2, -, x_n \in V$ and $a_1, a_2, -, a_n \in F$.

__Claim:__ T is linear

(Do yourself)

Note:- We will be using property 2, i.e.
[T is linear iff $T(cx+y) = cT(x)+T(y)$]
to prove a given transformation is linear.

Examples:-

(1) Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ as

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$$

Then T is linear.

→ Let $c \in \mathbb{R}$ and $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{R}^3$

then

$$T(c(a_1, a_2, a_3) + (b_1, b_2, b_3)) = T(ca_1+b_1, ca_2+b_2, ca_3+b_3)$$

$$= (ca_1+b_1 - (ca_2+b_2), 2(ca_3+b_3))$$

$$= (ca_1 - ca_2 + b_1 - b_2, 2ca_3 + 2b_3)$$

$$= (ca_1 - ca_2, 2ca_3) + (b_1 - b_2, 2b_3)$$

$$= c(a_1 - a_2, 2a_3) + (b_1 - b_2, 2b_3)$$

$$= cT(a_1, a_2, a_3) + T(b_1, b_2, b_3)$$

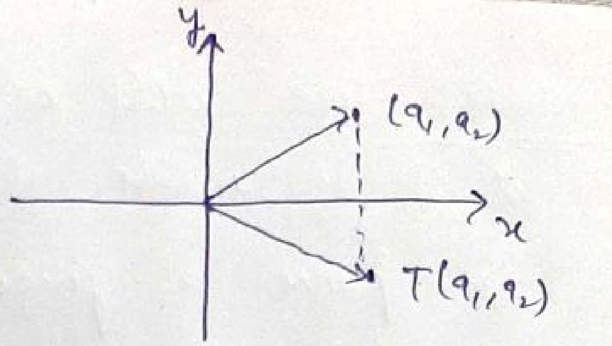
$\therefore T(c(a_1, a_2, a_3) + (b_1, b_2, b_3)) = cT(a_1, a_2, a_3) + T(b_1, b_2, b_3)$

$\forall$ $c \in \mathbb{R}$ and $(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathbb{R}^3$ are arb.

$\therefore$ T is linear transformation.

(2) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ as

$$T(a_1, a_2) = (a_1, -a_2)$$



is linear and called reflection about x-axis

→ Let $c \in \mathbb{R}$ and $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$

$$T(c(a_1, a_2) + (b_1, b_2)) = T((ca_1, ca_2) + (b_1, b_2))$$
$$= T[(ca_1 + b_1, ca_2 + b_2)]$$
$$= (ca_1 + b_1, -(ca_2 + b_2))$$
$$= (ca_1 + b_1, -ca_2 - b_2)$$
$$= (ca_1, -ca_2) + (b_1, -b_2)$$
$$= c(a_1, -a_2) + (b_1, -b_2)$$
$$= cT(a_1, a_2) + T(b_1, b_2)$$

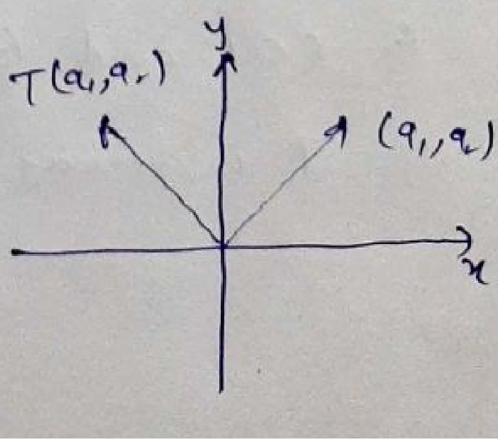
$\therefore T(c(a_1, a_2) + (b_1, b_2)) = cT(a_1, a_2) + T(b_1, b_2)$

As $c \in \mathbb{R}$ and $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$ are arb.

→ T is a linear transformation.

(3) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ as

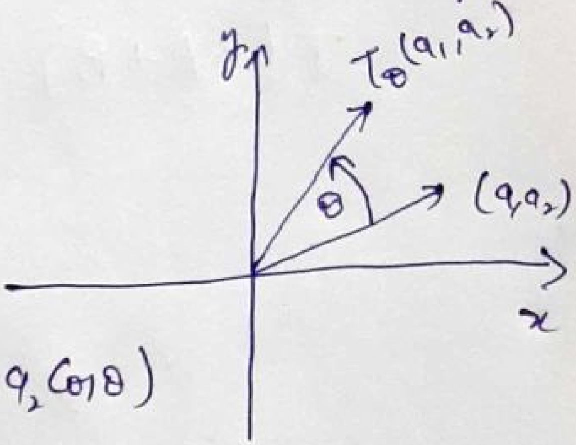
$$T(a_1, a_2) = (-a_1, a_2)$$



is linear and called reflection about y-axis

(Do yourself)

(4.) For any angle θ, Define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule: $T_\theta(a_1, a_2)$ is the vector obtained by rotating vector (a_1, a_2) counterclockwise by θ if $(a_1, a_2) \neq 0$ and $T_\theta(0,0) = 0$. This mapping is called <u>rotation</u> by θ.



Explicitly T_θ can be defined as

$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$$T_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta)$$

T_θ is linear transformation

$\rightarrow$ Let $c \in \mathbb{R}^2$ and $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$.

$$T_\theta(c(a_1, a_2) + (b_1, b_2)) = T_\theta(ca_1 + b_1, ca_2 + b_2)$$

$$= ((ca_1 + b_1)\cos\theta - (ca_2 + b_2)\sin\theta, (ca_1 + b_1)\sin\theta + (ca_2 + b_2)\cos\theta)$$

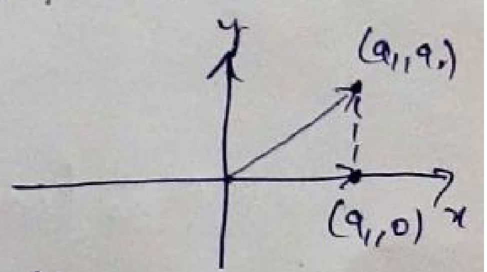
$$= (ca_1\cos\theta - ca_2\sin\theta, ca_1\sin\theta + ca_2\cos\theta)$$
$$+ (b_1\cos\theta - b_2\sin\theta, b_1\sin\theta + b_2\cos\theta)$$

$$= cT(a_1, a_2) + T(b_1, b_2)$$

$\therefore$ T is linear transformation.

(5.) Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(a_1, a_2) = (a_1, 0)$$



then T is linear and called projection on the x-axis (Do yourself).

(6.) Define $T : M_{m\times n}(F) \to M_{n\times m}(F)$ as

$$T(A) = A^t$$

then T is linear transformation

→ let $c \in F$ and $A, B \in M_{m\times n}(F)$

$$T(cA+B) = (cA+B)^t$$

$$= (cA)^t + B^t \quad (\because (A+B)^t = A^t + B^t)$$

$$= cA^t + B^t$$

$$= cT(A) + T(B)$$

$\therefore T(cA+B) = cT(A) + T(B)$

→ T is linear transformation.

(7.) Define $T : M_{n\times n}(F) \to F$ as

$$T(A) = \text{trace}(A)$$

then T is linear transformation

→ let $c \in F$ and $A, B \in M_{n\times n}(F)$

$$T(cA+B) = \text{trace}(cA+B)$$

$$= \text{trace}(cA) + \text{trace}(B)$$

$$= c\,\text{trace}(A) + \text{trace}(B)$$

$$= cT(A) + T(B)$$

→ T is linear transformation.

(8.) Let $V = C(\mathbb{R})$ = set of all continuous real valued functions on $\mathbb{R}$. Let $a, b \in \mathbb{R}$, $a < b$.
Define $T : V \to \mathbb{R}$ as
$$T(f) = \int_a^b f(t)\,dt$$
then T is linear transformation.

→ Let $c \in \mathbb{R}$ and $f, g \in V$.
$$T(cf+g) = \int_a^b (cf+g)(t)\,dt$$
$$= \int_a^b (cf(t)+g(t))\,dt$$
$$= \int_a^b cf(t)\,dt + \int_a^b g(t)\,dt$$
$$= c\int_a^b f(t)\,dt + \int_a^b g(t)\,dt$$
$$= cT(f) + T(g).$$

(9.) Let V be a Vector space.
then define $I_V : V \to V$ as
$$I_V(x) = x \quad , x \in V$$
is a linear transformation and called <u>Identity transformation</u> on V denoted by I_V.

(10.) Let V and W be vector spaces.
Define $T_0 : V \to W$ as

$$T_0(x) = 0 \quad ; x \in V.$$

Then T_0 is linear transformation and is called <u>zero transformation</u>.

<u>Defn</u>:- Let V and W be vector spaces and let $T: V \to W$ be linear. We define the <u>null space</u> (or <u>kernel</u>) denoted by $N(T)$, of T to be the set of all vectors x in V s.t. $T(x) = 0$.

i.e. $N(T) = \{x \in V : T(x) = 0\}$.

We define <u>Range</u> (or <u>image</u>) denoted by $R(T)$, of T to be the subset of W as

$$R(T) = \{T(x) : x \in V\}.$$

<u>Note</u>:- $N(T) \subseteq V$ and $R(T) \subseteq W$.

<u>Examples</u>:-

(1.) Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$$

then T is a linear transformation (show).

$$N(T) = \{(a_1, a_2) \in \mathbb{R}^2 : T(a_1, a_2) = (0, 0, 0)\}.$$

As $T(a_1, a_2) = (0, 0, 0)$

$\Rightarrow (a_1+a_2, 0, 2a_1-a_2) = (0,0,0)$

$\Rightarrow \quad a_1+a_2=0$

$\& \quad 2a_1-a_2=0$

Solving these we get $a_1=0$, $a_2=0$.

$\therefore N(T) = \{(0,0)\}$

and $R(T) = \{T(x) : x \in \mathbb{R}^2\}$

$= \{(a_1+a_2, 0, 2a_1-a_2) : (a_1,a_2) \in \mathbb{R}^2\}$

$\Rightarrow \quad R(T) = \{(x,0,y) : \left(\frac{x+y}{3}, \frac{2x-y}{3}\right) \in \mathbb{R}^2\}$

(2.) Let T be the reflection about x-axis.

i.e. $T: \mathbb{R}^2 \to \mathbb{R}^2$ as

$T(a_1,a_2) = (a_1,-a_2)$

then $N(T) = \{(a_1,a_2) \in \mathbb{R}^2 : T(a_1,a_2) = (0,0)\}$

$= \{(a_1,a_2) \in \mathbb{R}^2 : (a_1,-a_2) = (0,0)\}$

$= \{(0,0)\}$

and $R(T) = \{T(a_1,a_2) : (a_1,a_2) \in \mathbb{R}^2\}$

$= \{(a_1,-a_2) : (a_1,a_2) \in \mathbb{R}^2\}$

$= \{(a_1,a_2) : (a_1,-a_2) \in \mathbb{R}^2\}$

$R(T) = \mathbb{R}^2$.