Conjugacy in S_n:-

In this section we consider conjugation of symmetric group.

Def^n: Cycle-type:- If $\sigma \in S_n$ is the product of disjoint cycles of length $n_1, n_2, \ldots, n_r$ with $n_1 \leq n_2 \leq \ldots \leq n_r$ (including 1-cycles) then the integers $n_1, n_2, \ldots, n_r$ are called the cycle type of σ.

Examples (1). Let $\sigma = (1\,3)(2\,4\,6) \in S_{10}$

then its cycle type is $1,1,1,1,1,2,3$.

(2) Let $\sigma = (1\,2\,3)(5\,6)(7\,8) \in S_8$

then its cycle type is $1,2,2,3$.

Partition:- If $n \in \mathbb{Z}^+$, a partition of n is any nondecreasing sequence of positive integers whose sum is n.

Note:- In S_n, the cycle type of any permutation is partition of n.

Proposition (10):- Let σ, τ be elements of S_n and suppose that σ has cycle decomposition

$$(a_1\, a_2 \ldots a_{k_1})(b_1\, b_2 \ldots b_{k_2}) \ldots$$

Then $\tau\sigma\tau^{-1}$ has cycle decomposition

$(\tau(a_1)\ \tau(a_2) \dots \tau(a_{k_1}))(\tau(b_1)\ \tau(b_2) \dots \tau(b_{k_2}))\dots$

that is, $\tau\sigma\tau^{-1}$ is obtained from σ by replacing each entry i in the cycle decomposition for σ by the entry $\tau(i)$.

<u>Example</u>

(1.) $\sigma = (12)(345)(6789)$ and $\tau = (1357)(2468)$.

then $a_1 = 1 \quad a_2 = 2$

$b_1 = 3 \quad b_2 = 4 \quad b_3 = 5$

$c_1 = 6 \quad c_2 = 7 \quad c_3 = 8 \quad c_4 = 9$

Then $\tau\sigma\tau^{-1} = (\tau(a_1)\ \tau(a_2))(\tau(b_1)\ \tau(b_2)\ \tau(b_3))(\tau(c_1)\ \tau(c_2)\ \tau(c_3)\ \tau(c_4))$

$= (34)(567)(8129)$.

(2.) $\sigma = (135)(24)$ and $\tau = (12345)$ in S_5.

then

$\tau\sigma\tau^{-1} = (241)(35)$.

<u>Note!-</u> σ and $\tau\sigma\tau^{-1}$ has the ~~cy~~ same cycle type.

<u>Proof!-</u> observe that $\sigma(a_1) = a_2$

then $\tau\sigma\tau^{-1}(\tau(a_1)) = \tau\sigma(\tau^{-1}\tau(a_1))$

$= \tau\sigma(a_1)$

$= \tau(a_2)$

In general if $\sigma(i) = j$

then $\tau\sigma\tau^{-1}(\sigma(i)) = \tau(j)$

Thus, if the ordered pair i, j appears in the cycle decomposition of σ, then the ordered pair $\tau(i), \tau(j)$ appears in the cycle decomposition of $\tau\sigma\tau^{-1}$ and is in the same position as of i, j.

Proposition 11 :- Two elements of S_n are conjugate in S_n if and only if they have same cycle type. The no. of conjugacy classes in S_n equals the number of partitions of n.

Proof :- Let σ_1 and σ_2 be two elements in S_n and let σ_1 is conjugate of σ_2.

then there exists $\tau \in S_n$ s.t. $\sigma_1 = \tau\sigma_2\tau^{-1}$

Then By Proposition 10, σ_1 and σ_2 have same cycle type.

Conversely, let σ_1 and σ_2 has same cycle type. Order the cycles in nondecreasing length, including 1-cycles.

Suppose $\sigma_1 = (a_1 \dots a_{k_1})(b_1 \dots b_{k_2}) \dots$

$\sigma_2 = (a_1' \dots a_{k_1}')(b_1' \dots b_{k_2}')$

where $k_1 \le k_2 \le \dots$

Define τ to be the function which maps ith integer (ignoring parenthesis) in the list for σ_1 to the ith integer in the list for σ_2.

Thus τ is a permutation

and $\tau\sigma_1\tau^{-1} = \sigma_2$

$\Rightarrow$ σ_1 and σ_2 are conjugate.

See for a_1',
$\sigma_2(a_1') = a_2'$
& $\tau\sigma_1\tau^{-1}(a_1')$
$= \tau\sigma_1(a_1) = \tau(a_2)$
$= a_2'$

Since each cycle type for a permutation in S_n is a partition of n.

$\therefore$ There is a bijection between cycle types ~~for a~~ and partition of n.

Also There is bijection between the conjugacy classes of S_n and the cycle types.

Therefore there is bijection between conjugacy classes of S_n and partition of n.

$\Rightarrow$ The no. of conjugacy classes in S_n equals the number of partitions of n.

Ex 1 let $\sigma_1 = (1)(3\,5)(8\,9)(2\,4\,7\,6)$

and $\sigma_2 = (3)(4\,7)(8\,1)(5\,2\,6\,9)$

then $\tau = (1\,3\,4\,2\,5\,7\,6\,9)(8)$

s.t. $\tau\sigma_1\tau^{-1} = \sigma_2$

(2) If $n=4$, the partitions of ~~5~~ 4 and corresponding representatives of conjugacy classes

Partition of 4	Representative of Conjugacy cl.
1, 1, 1, 1	(1)
1, 1, 2	(12)
1, 3	(123)
2, 2	(12)(34)
4	(1234)

(3.) If $n=5$, the partitions of 5 and corresponding representatives of conjugacy classes

Partition of 5	Representative of conjugacy class
1, 1, 1, 1, 1	(1)
1, 1, 1, 2	(12)
1, 1, 3	(123)
1, 4	(1234)
1, 2, 2	(12)(34)
2, 3	(12)(345)
5	(12345)

Centralizers of an element in S_n

Let σ be an m-cycle in S_n, then the total no. of m-cycles in S_n is

$$\frac{n.(n-1)\ldots(n-m+1)}{m}$$

$\Rightarrow$ The no. of conjugates of $\sigma = \dfrac{n.(n-1)\ldots(n-m+1)}{m}$

and The no. of conjugates of $\sigma = |S_n : C_{S_n}(\sigma)|$

$$\therefore \quad \frac{|S_n|}{|C_{S_n}(\sigma)|} = \frac{n.(n-1)\ldots(n-m+1)}{m}$$

$$\Rightarrow \quad |C_{S_n}(\sigma)| = \frac{n!\,.\,m}{n.(n-1)\ldots(n-m+1)}$$

$$|C_{S_n}(\sigma)| = m.(n-m)!$$

and $\quad C_{S_n}(\sigma) = \{\sigma^i \tau \mid 0 \le i \le m-1, \tau \in S_{n-m}\}$

where S_{n-m} denotes the subgp of S_n which fixes all integers appearing in the σ.

Propⁿ:- Normal subgroups of a group G are the union of conjugacy classes of G, i.e. if $H \trianglelefteq G$, then for every conjugacy class K of G either $K \subseteq H$ or $K \cap H = \phi$.

Proof:- let $x \in K \cap H$

then $x \in K$ and $x \in H$

then $gxg^{-1} \in gHg^{-1}$ $\forall g \in G$.

And as H is normal, $\Rightarrow$ $gHg^{-1} = H$ $\forall g \in G$.

$\therefore$ $gxg^{-1} \in H$.

and $gxg^{-1} \in K$

$\Rightarrow$ $K \subseteq H$.

Right Group Action:-

Right group action of the group G on the nonempty set A is a map from $A \times G$ to A denoted by $a \cdot g$ for $a \in A$ and $g \in G$, that satisfies following

(1) $(a \cdot g_1) \cdot g_2 = a \cdot (g_1 g_2)$ for all $a \in A$, and $g_1, g_2 \in G$.

(2) $a \cdot 1 = a$ for all $a \in A$.