~~Example:~~ A pensioner receives 2000 \$ at the beginning of each month. The amount of money he needs to spend during a month is independent of the amount he has and is equal to i with probability P_i $(i=1,2,3,4)$ & $\sum P_i = 1$. If the pensioner has more than 3000 \$ at the end of a month, he gives the amount greater than 3000 \$ to his son. If, after receiving his payment at the beginning of a month, the pensioner has a capital of 5000, what is the probability that his capital is ever 1000 or less at any time within the following four months.

Sol^n Consider Markov chain with states {1,2,3}

and transition probability matrix $Q = [Q_{ij}]$

given by

$$Q = \begin{array}{cc} & \begin{array}{ccc} (\text{less than } 1000\$) & (1000-2000) & (2000-3000) \\ 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ P_3+P_4 & P_2 & P_1 \\ P_4 & P_3 & P_1+P_2 \end{bmatrix} \end{array}$$

Q_{21} is the probability that a month that ends with the pensioner having the amount 2000 will be followed by a month that ends with the pensioner having less than or equal to 1000\$.

$\because$ Pensioner receives 2000\$ at the beginning

$\therefore$ amount in new month = 2000 + 2000 = 4000\$

his ending capital will be less than or

equal to 1 if his expenses are either 3000\$ or 4000\$.

Thus, $Q_{21} = P_3 + P_4$

In particular, if $P_i = \frac{1}{4}$, $i = 1, 2, 3, 4$.

The transition probability matrix is

$$Q = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 1/2 & 1/4 & 1/4 \\ 3 & 1/4 & 1/4 & 1/2 \end{array}$$

$$Q^{(4)}_{3,1} = ?$$

$$Q^2 = \begin{bmatrix} 1 & 0 & 0 \\ 7/16 & 1/8 & 7/16 \\ 1/2 & 3/16 & 5/16 \end{bmatrix}$$

$$Q^4 = Q^2 \cdot Q^2 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 1 & 0 & 0 \end{array}$$

$$P^4 = P^2 \cdot P^2 = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ \frac{222}{256} & \frac{13}{256} & \frac{21}{256} \\ \frac{201}{256} & \frac{21}{256} & \frac{34}{256} \end{bmatrix}$$

$$\text{required probability} = P_{31}^{(4)} = \frac{201}{256}.$$