Theorem: Let $V$ and $W$ be vector spaces over $F$, and suppose that $\{v_1, v_2, \dots, v_n\}$ is a basis for $V$. For $w_1, w_2, \dots, w_n$ in $W$, there exists exactly one linear transformation $T: V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \dots, n$

Proof: Let $x \in V$. Then

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$$x = \sum_{i=1}^{n} a_i v_i$$

$\left[ \because \{v_1, v_2, \dots, v_n\} \text{ is a basis for } V \right.$

where $a_1, a_2, \dots, a_n$ are unique scalars.

Now Define $T: V \to W$ by

$$T(x) = \sum_{i=1}^{n} a_i w_i \qquad \text{——— } (1)$$

We claim that $T(v_i) = w_i \;\forall\; i = 1, 2, \dots n$

and $T$ is linear transformation.

$$v_i = 0.v_1 + 0.v_2 + \cdots + 1.v_i + 0.v_{i+1} + \cdots + 0.v_n$$

Now

from eqn (1),

$$T(v_i) = 0.w_1 + 0.w_2 + \cdots + 1.w_i + 0.w_{i+1} + \cdots + 0.w_n$$

$$\Rightarrow T(v_i) = w_i \quad \forall \quad i = 1, 2, \ldots, n.$$

~~T is linear~~

Let $u, v \in V$ and $\alpha \in F$.

Then $u = \sum_{i=1}^{n} b_i v_i$ & $v = \sum_{i=1}^{n} c_i v_i$

where $b_i$'s & $c_i$'s are scalars, $1 \le i \le n$.

Now $$T(\alpha u + v) = T\left(\alpha \sum_{i=1}^{n} b_i v_i + \sum_{i=1}^{n} c_i v_i\right)$$

$$= T\left(\sum_{i=1}^{n} \alpha b_i v_i + \sum_{i=1}^{n} c_i v_i\right)$$

$$= T\left(\sum_{i=1}^{n} (\alpha b_i + c_i) v_i\right)$$

$$= T\left(\sum_{i=1}^{n} (\alpha b_i + c_i) v_i\right)$$

$$= \sum_{i=1}^{n} (\alpha b_i + c_i) w_i \quad \left[\text{using eq}^n \text{ (1)}\right]$$

$$= \sum_{i=1}^{n} \alpha b_i w_i + \sum_{i=1}^{n} c_i w_i$$

$$= \alpha \sum_{i=1}^{n} b_i w_i + \sum_{i=1}^{n} c_i w_i$$

$$= \alpha \left\{ T\left(\sum_{i=1}^{n} b_i v_i\right) \right\} + T\left(\sum_{i=1}^{n} c_i v_i\right)$$

$$\left[\text{using eq}^n \text{ (1)}\right]$$

$$= \alpha T(u) + T(v)$$

Thus, $T$ is linear.

Now we will show that this linear transformation $T$ is unique.

Suppose that $U : V \to W$ is linear

& $U(v_i) = w_i$ for $i = 1, 2, \ldots, n$

then for $x \in V$ with $x = \sum_{i=1}^{n} a_i v_i$

we have
$$U(x) = U\left(\sum_{i=1}^{n} a_i v_i\right)$$
$$= \sum_{i=1}^{n} a_i U(v_i) \quad \left\{ \because U \text{ is linear} \right\}$$
$$= \sum_{i=1}^{n} a_i w_i$$
$$= T(x)$$

Hence, $U = T$.

Thus, $T$ is unique.

~~Corollary~~. Let $V$ and $W$ be vector spaces and suppose that $V$ has a finite basis $\{v_1, v_2, \dots, v_n\}$.
If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \dots, n$ then $U = T$.
i.e. $U(x) = T(x)$ $\forall x \in V$.