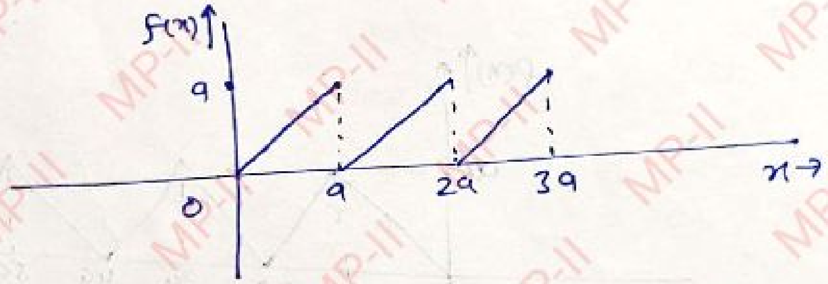
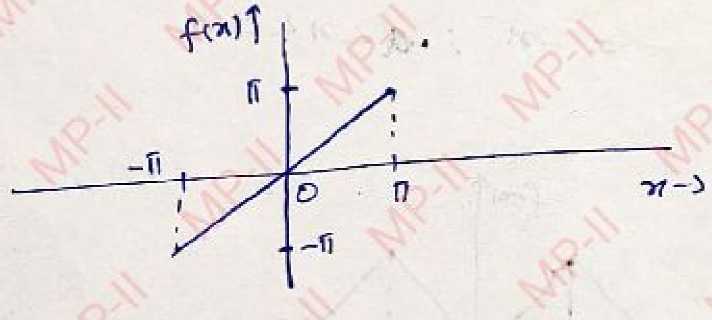
① Sawtooth wave function : The sawtooth wave function with a period 'a' is defined as

$$f(x) = x \quad \text{for } 0 < x < a$$
$$= 0 \quad \text{for } x \le 0 \qquad \& \ f(a+x) = f(x)$$

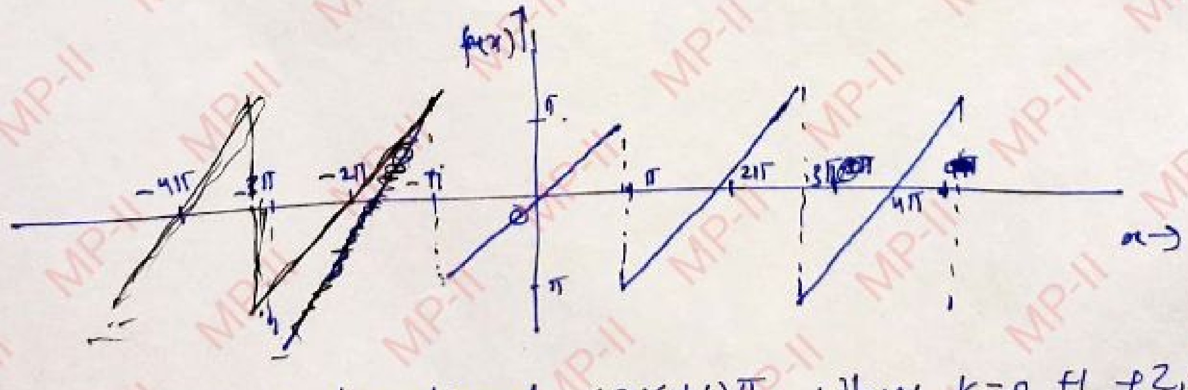


& Sawtooth wave function with period 2π is defined as

$$f(x) = x \quad \text{for } -\pi < x < \pi$$
$$= 0 \quad \text{elsewhere} \qquad ; \ f(x+2\pi) = f(x)$$



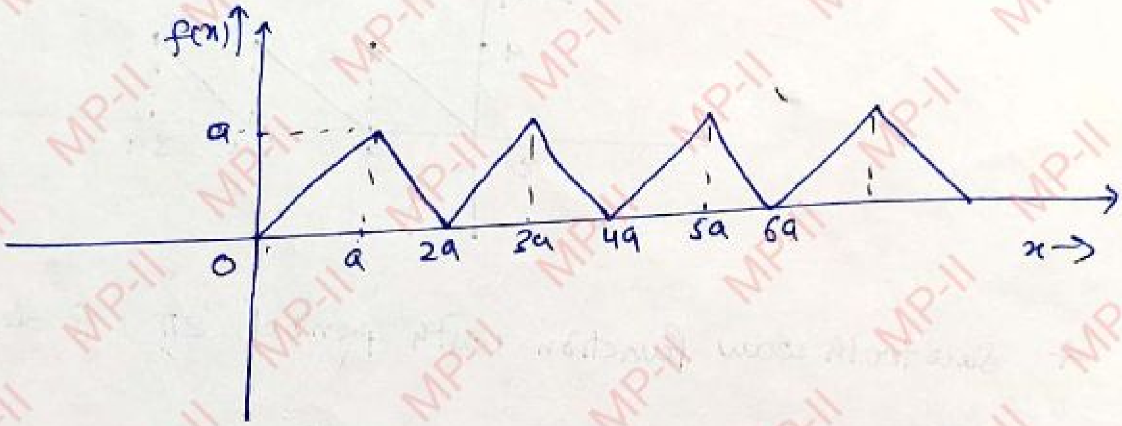
& the graph of extended periodic function with period 2π is



It has discontinuities at $(2K+1)\pi$, where $K = 0, \pm1, \pm2, \pm3 \ldots$

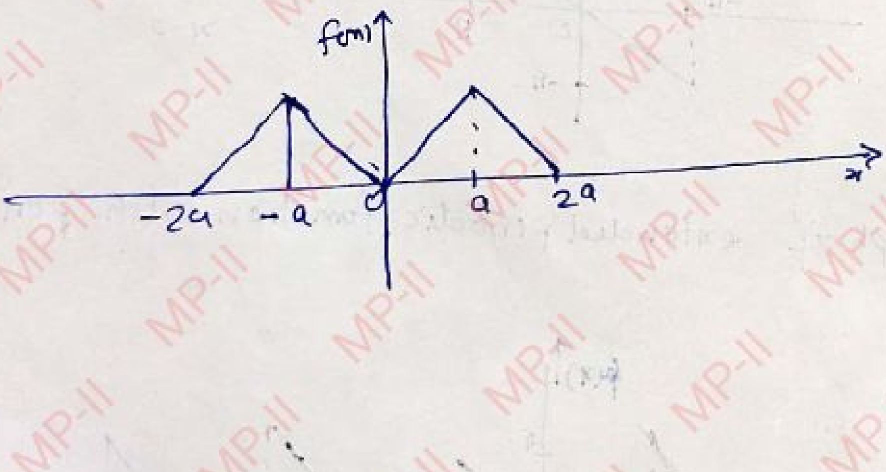
② Triangular wave function : The triangular wave function $f(x)$ with a period $2a$ is defined as

$$f(x) = x \quad \text{for } 0 < x < a$$
$$= 2a - x \quad \text{for } a < x < 2a$$
$$\& \quad f(x+2a) = f(x)$$



&

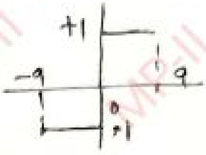
$$f(x) = x \quad ; \; 0 < x < a \qquad \text{with } 2a$$
$$= -x \quad ; \; -a < x < 0$$



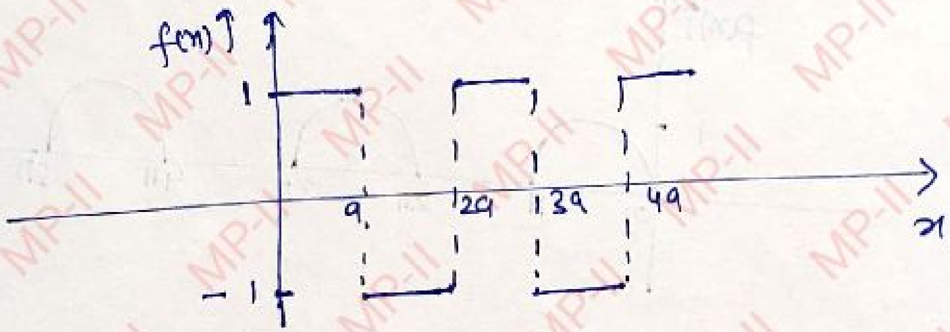
(3) Square wave function :

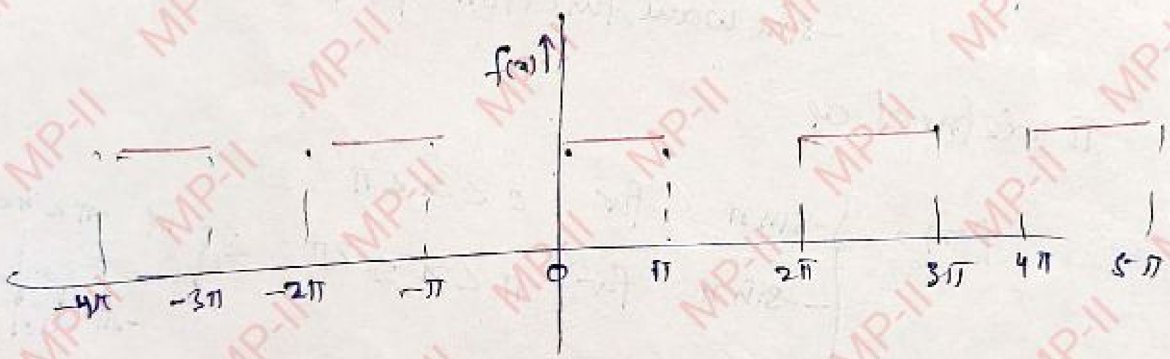
The square wave function $f(x)$ with a period $2a$ is defined as

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < a \\ -1 & \text{for } a < x < 2a \end{cases} \Rightarrow -a < x < 0$$



$\&$ $f(x+2a) = f(x)$





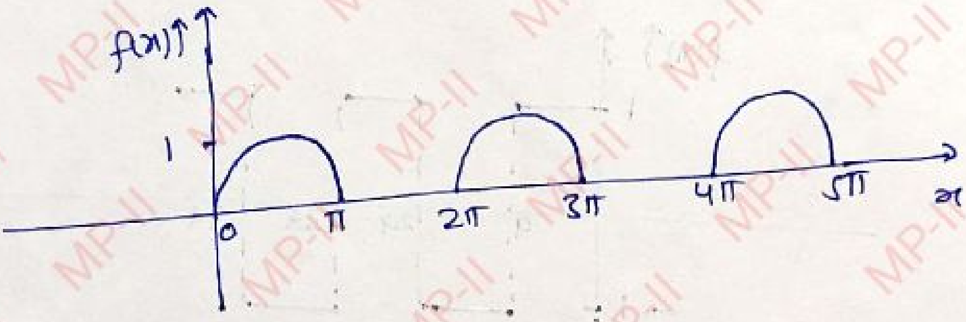
$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ a & \text{if } 0 \leq x < \pi \end{cases}$$

(4) Half wave Rectified Sinusoidal function ; H·W·R·S·F

$f(x)$ with a period 2π is defined as

$$f(x) = \begin{cases} \sin x & \text{for } 0 < x < \pi \\ 0 & \text{for } \pi < x < 2\pi \end{cases} \Rightarrow \{ \text{if } -\pi < x < 0$$

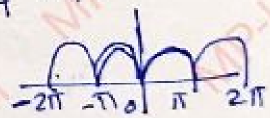
$$f(x + 2\pi) = f(x)$$



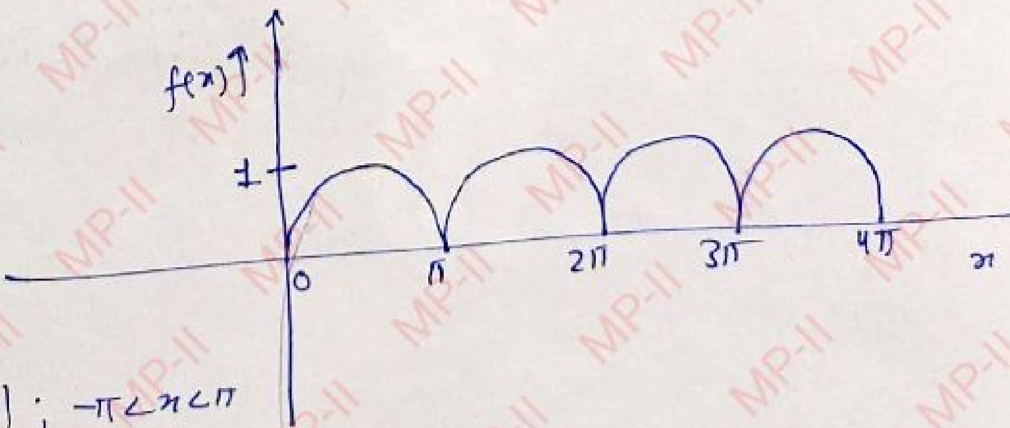
(5) Full wave Rectified wave function :

The full wave rectified sine wave function $f(x)$ with a period 2π is defined as

$$f(x) = \begin{cases} \sin x & \text{for } 0 < x < \pi \\ -\sin x & \text{for } \pi < x \le 2\pi \end{cases} \Rightarrow \{ \text{if } -\pi < x < 0$$

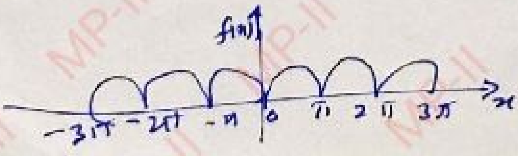


& $f(x + 2\pi) = f(x)$



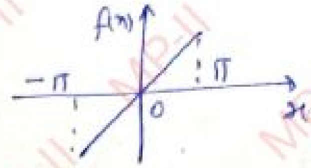
OR

$f(x) = |\sin x| \; ; \; -\pi < x < \pi$



Que: 1 Determine the fourier series for Sawtooth function $f(x)$ defined by $f(x) = x \;,\; -\pi < x < \pi$

$= 0$ otherwise



Sol: The fourier series expansion of $f(x) = x$ be

$$f(x) = x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad --- (1)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$ [$\because$ gt odd function]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx = 0 \quad \left[\text{again odd function}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx\,dx = \frac{2}{\pi}\left[x\cdot\left(-\frac{\cos nx}{n}\right)\Big|_{0}^{\pi} + \int_{0}^{\pi} 1\,\frac{\cos nx}{n}\,dx \right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n} + 0 + \frac{\sin nx}{n^2}\Big|_{0}^{\pi}\right]$$

$$= -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$\therefore f(x) = x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

$$\boxed{x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)}$$

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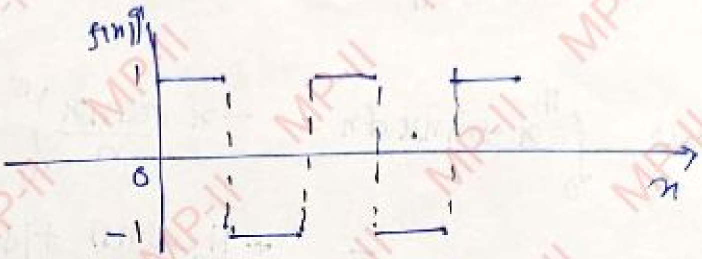
Que (3) Find Fourier series expansion of a square wave function

Square Wave function

$$f(x) = 1 \quad \text{for } 0 < x < \pi$$
$$= -1 \quad \text{for } \pi < x < 2\pi \text{ or } -\pi < x < 0$$

Sol :-



The fourier series expansion of $f(x)$ be.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{———} (1)$$

where
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_0^{\pi} 1.dx - \int_{\pi}^{2\pi} 1\,dx\right]$$
$$= \frac{1}{2\pi}\left[x\Big)_0^{\pi} - x\Big)_{\pi}^{2\pi} \right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}\cos nx\,dx + \int_{\pi}^{2\pi} -\cos nx\,dx\right]$$
$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\Big)_0^{\pi} - \frac{\sin nx}{n}\Big)_{\pi}^{2\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_0^{\pi}\sin nx\,dx - \int_{\pi}^{2\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{\cos nx}{n}\Big)_0^{\pi} + \frac{\cos nx}{n}\Big)_{\pi}^{2\pi}\right] = \frac{1}{n\pi}\left[-(-1)^n + 1 + (-1)^{2n} - (-1)^n\right]$$
$$= \frac{1}{n\pi}\left[(-1)^{2n} - 2(-1)^n + 1\right]$$

$$= \frac{1}{\pi}\left[\cancel{-\frac{\cos n\pi}{n} + \frac{1}{n} + \frac{\cos 2n\pi}{n} - \frac{\cos n\pi}{n}}\right]$$

$$= \frac{1}{\pi}\left[\cancel{-\frac{(-1)^n}{n} + \frac{1}{n} + \frac{1}{n}(-1)^{2n} - \frac{(-1)^n}{n}}\right]$$

$$= \frac{1}{\pi}\left[\cancel{\frac{1}{n} + \frac{2(-1)^{n+1}}{n} + \frac{1}{n}(-1)^{2n}}\right] = 0 \quad \text{for } n = \text{even}$$
$$= \frac{4}{n\pi} \quad \text{for } n = \text{odd}$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

$$\boxed{f(x) = \frac{4}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]}$$

App

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} K \sin nx \, dx \quad \text{, here } K = \text{any no.}$$

$$= \frac{2K}{\pi n}\left[-\cos nx\right]_0^{\pi} = 0 \text{ for } n = \text{even}$$

$$= \frac{4K}{n\pi} \text{ for } n = \text{odd}$$

$$f(x) = \frac{1}{\pi}\left[\int_{-\pi}^{0} -K \sin nx \, dx + \int_{0}^{\pi} K \sin nx \, dx\right]$$

$$= \frac{1}{\pi}\left[\left[+K\frac{\cos nx}{n}\right]_{-\pi}^{0} - \left(K\frac{\cos nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{K}{\pi n}\left[\cancel{\frac{K}{n}(1+1) - \frac{K}{n}(1-1)}\right]\left[1 - \cos n\pi - \cos n\pi + 1\right]$$

$$= \frac{2K}{\pi n}\left[1 - (-1)^n\right] = 0 \text{ for } n = \text{even}$$

$$= \frac{4K}{\pi n} \text{ for } n = \text{odd}$$

(4) Half Wave Rectifier :-

$$f(x) = \sin x \quad \text{for } 0 < x < \pi$$
$$= 0 \quad \text{for } \pi < x < 2\pi$$

& deduce (i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \cdots = \frac{1}{2}$

(ii) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} = \frac{\pi - 2}{4}$

Sol : $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin x\,dx =$

$$= \frac{1}{2\pi}\Big[-\cos x\Big]_0^{\pi} = \frac{1}{2\pi}[1+1] = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\sin x \cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \quad **$$

$$= -\frac{1}{2\pi}\left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \Rightarrow -\frac{1}{\pi(n^2-1)}[(-1)^n+1]$$

$$= \begin{cases} 0 & \text{when } n = \text{odd} \\ -\frac{2}{\pi(n^2-1)} & \text{when } n = \text{even} \end{cases}$$

Ex ** see proof at last

$$= -\frac{1}{2\pi}\left[\frac{\cos(n+1)\pi - 1}{n+1} - \frac{\cos(n-1)\pi - 1}{n-1}\right] = -\frac{1}{2\pi}\left[\frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(-1)^{n+2}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right]$$

$$= -\frac{1}{\pi(n^2-1)}[(-1)^n+1]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\,dx = \frac{1}{2\pi}\int_0^{\pi} 2\sin x \sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right]dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$= 0$ for $n \neq 1$

$= \frac{1}{2}$ for $n = 1$ using ①

OR

$$\frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx = \frac{1}{\pi}\int_0^{\pi}\frac{(1-\cos 2x)}{2}dx$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}\right) - \frac{1}{\pi}\int_0^{\pi}\cos 2x\,dx$$

$$= \frac{1}{2}$$

Hence required fourier series is

$$f(x) = \frac{1}{\pi} + \sum_{n=\text{even}}^{\infty} -\frac{2}{\pi(n^2-1)}\cos nx + \sum_{n=1} \frac{1}{2}\sin nx$$

$$f(x) = \frac{1}{\pi} - \frac{2}{3\pi}\cos 2x - \frac{2}{15\pi}\cos 4x - \frac{2}{35\pi}\cos 6x + \cdots + \frac{1}{2}\sin x \quad (A)$$

(i) put $x = \pi$ in (A)

$$0 = \frac{1}{\pi}\left[1 - \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} - \frac{2}{5\cdot 7} - \cdots\right] = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots\right)$$

or $$\boxed{\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} = \frac{1}{2}}$$

$\frac{1}{2}\sin x = \frac{1}{2}$

(ii) put $x = \frac{\pi}{2}$; $$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{1}{\pi}\left(\frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots\right)$$

$$\pi = \left[\frac{2+\pi}{2} + 2(\quad)\right]$$

$$\pi - \frac{2+\pi}{2} = 2(\quad)$$

$$\frac{2\pi - \pi - 2}{2} = 2(\quad)$$

$$\frac{\pi-2}{4} = (\quad)$$

$$\Rightarrow \boxed{\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots}$$

(5) Full Wave Rectifier :

$f(x) = |\sin x|\ ;\ -\pi < x < \pi$

OR

$f(x) = \sin x\ ;\ 0 < x < \pi$

$\quad = -\sin x\ ;\ -\pi < x < 0$

or $\pi < x < 2\pi$

Fourier series expansion of f(x) be

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---} (1)$$

where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\sin x\,dx + \int_{0}^{\pi} \sin x\,dx\right]$

$$= \frac{1}{\pi}\left[(\cos x)_{-\pi}^{0} + (-\cos x)_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[1-(-1) - (-1-1)\right]$$

$$= \frac{1}{2\pi}\left[1+1+1+1\right] = \frac{4}{2\pi} = \frac{2}{\pi}$$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin x\cos nx\,dx + \int_{0}^{\pi} \sin x\cos nx\,dx\right]$

Now $\frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\,dx = \frac{1}{2\pi}\int_{0}^{\pi} 2\underset{B}{\sin x}\,\underset{A}{\cos nx}\,dx$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= -\frac{1}{2\pi}\left[+\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}\left[\frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right]$$

$$= \begin{cases} 0 & \text{when } n = \text{odd} \\ -\frac{2}{\pi(n^2-1)} & \text{when } n = \text{even} \end{cases}$$

similarly

$$-\frac{1}{\pi}\int_{-\pi}^{0} \sin x \cos nx\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2\sin x\cos nx\,dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} \left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= -\frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_{-\pi}^{0}$$

$$= -\frac{1}{2\pi}\left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_{0}^{-\pi}$$

$$= -\frac{1}{2\pi}\left[\frac{(-1)^{n+1}-1}{n+1} - \frac{(-1)^{n-1}-1}{n-1}\right]$$

$$= \begin{cases} 0 & \text{when } n = \text{odd} \\ -\frac{2}{\pi(n^2-1)} & \text{when } n = \text{even} \end{cases}$$

$\therefore$ $a_n = -\frac{4}{\pi(n^2-1)}$ when n = even

$= 0$ when n = odd

Now $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin x \sin nx\,dx + \int_{0}^{\pi} \sin x \sin nx\,dx\right]$

Now $\frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\, dx = \frac{1}{2\pi}\int_0^{\pi} 2\sin x \sin nx\, dx$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[0\right] = 0$$

Similarly $-\frac{1}{\pi}\int_{-\pi}^{0} \sin x \sin nx\, dx = 0$.

So we have

$$f(x) = \frac{2}{\pi} + \sum_{n=\text{even}}^{\infty} -\frac{4}{\pi(n^2-1)}\cos nx$$

$$\boxed{f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right)}$$