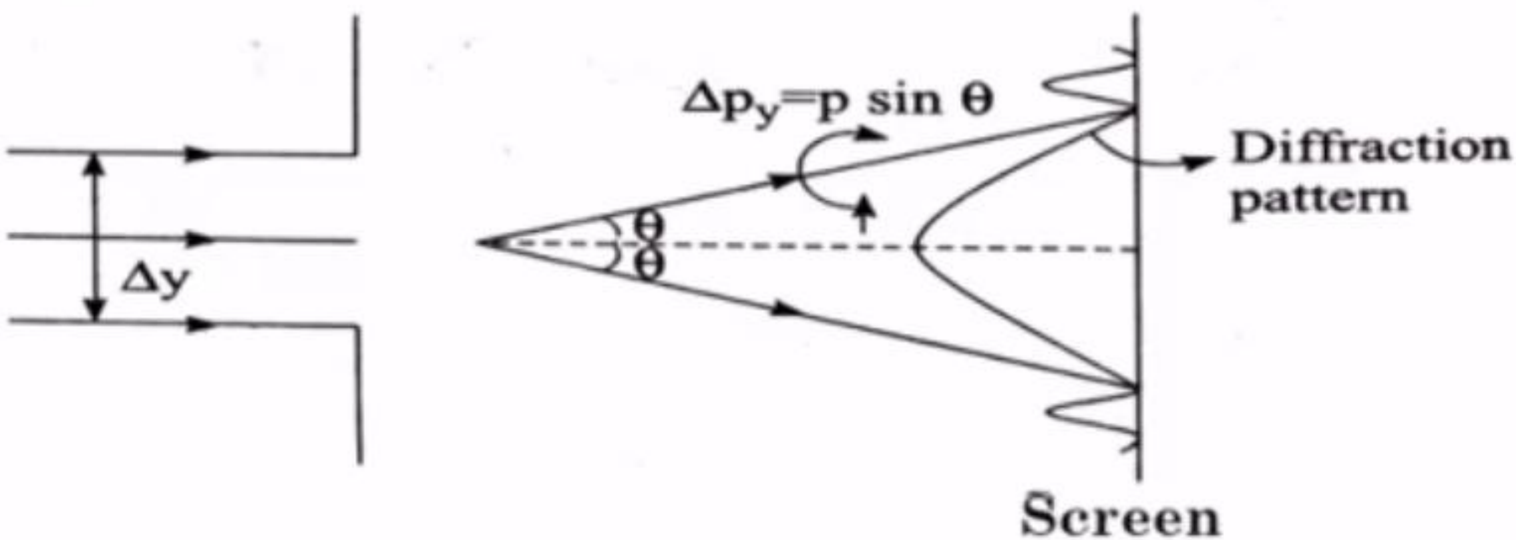
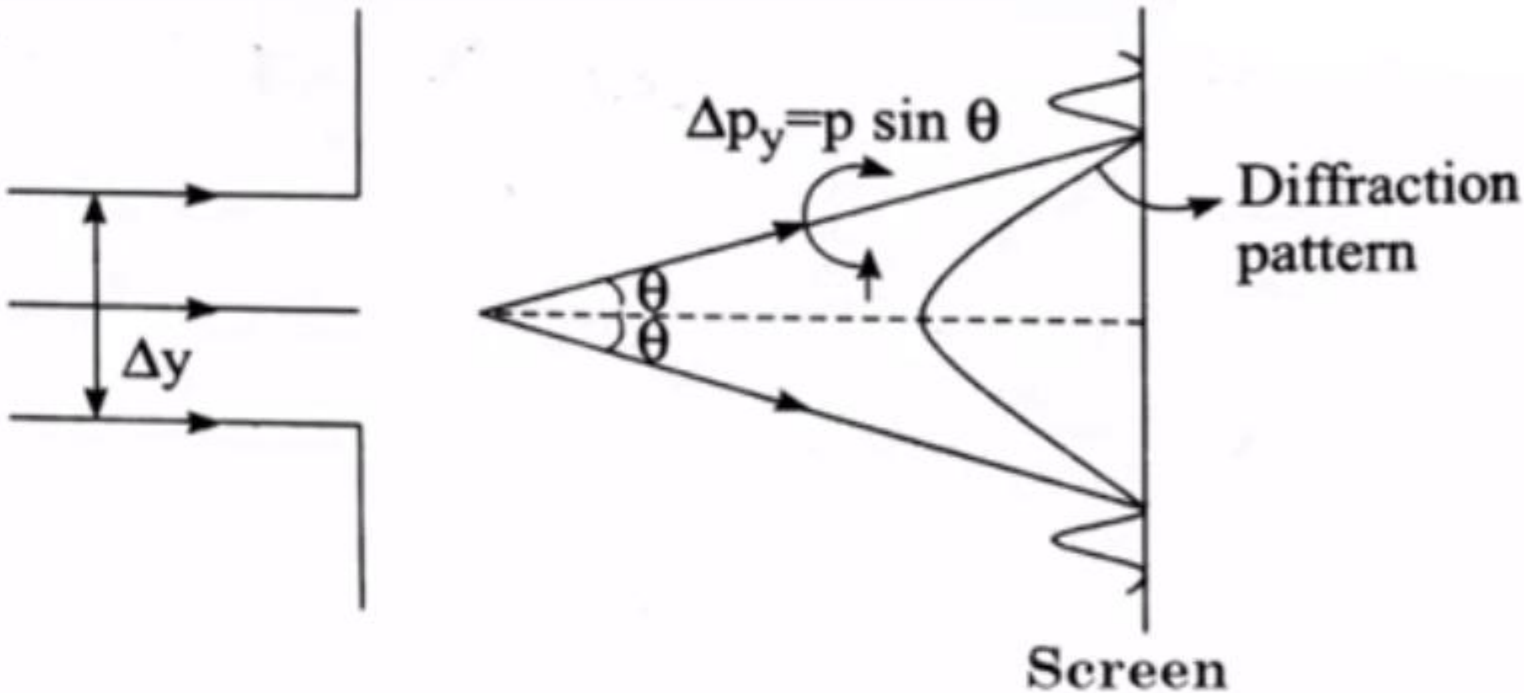
Applications of Uncertainty Principle

Diffraction by a single slit :

As we have seen the microscopic particles like electrons show wave behaviour and produce diffraction pattern. Consider a narrow beam of electrons passing through a single slit and diffraction pattern be observed on the screen.

In producing diffraction pattern all the electrons in the beam must pass-through the slit. In case of microscopic particle we can not follow the particle along its motion nor we can determine its path. Therefore the position which the electron has passes at the slit is uncertain. The amount of uncertainty in the measurement of position is Δy, the width of the slit.





From the theory of diffraction for first order ($n = 1$) we have

$$d \sin\theta = n\lambda$$

Here in this case $d = \Delta y$ and $n = 1$ so that,

$$\Delta y . \sin\theta = \lambda$$

or

$$\Delta y = \frac{\lambda}{\sin\theta} \qquad \ldots(1)$$

Initially electrons are moving along the x-axis and have not component of momentum along y-axis $p\sin\theta$. This component may have value in the range from $p\sin\theta$ to $-p\sin\theta$ after diffraction. Therefore uncertainty in y component of momentum will be by

$$\Delta p_y = 2p\sin\theta = 2\frac{h}{\lambda}.\sin\theta \qquad \ldots(2)$$

$$\left(\because \quad p = \frac{h}{\lambda}\right)$$

Hence the product of uncertainties in position and momentum from equations (1) and (2) is

$$\Delta y.\Delta P_y = \frac{\lambda}{\sin\theta}.\frac{2h\sin\theta}{\lambda} = 2h$$

$$\Delta y.\Delta p_y \geq \frac{\hbar}{2}$$

This is uncertainty relation.

Initially electrons are moving along the x-axis and have not component of momentum along y-axis $p\sin\theta$. This component may have value in the range from $p\sin\theta$ to $-p\sin\theta$ after diffraction. Therefore uncertainty in y component of momentum will be by

$$\Delta p_y = 2p\sin\theta = 2\frac{h}{\lambda}.\sin\theta \qquad \ldots(2)$$

$$\left(\therefore \quad p = \frac{h}{\lambda}\right)$$

Hence the product of uncertainties in position and momentum from equations (1) and (2) is

$$\Delta y.\Delta P_y = \frac{\lambda}{\sin\theta}.\frac{2h\sin\theta}{\lambda} = 2h$$

$$\Delta y.\Delta p_y \geq \frac{\hbar}{2}$$

This is uncertainty relation.

The Ground State Energy and the Radius of the Hydrogen Atom

The position-momentum uncertainty relation can be used to obtain an estimate of the energy and the radius of an atom in its ground state. Let us discuss the simplest atom, hydrogen, which consists of a proton and an electron. If we assume the proton to be infinitely heavy, and hence at rest, the total classical energy of the electron is given by

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r}$$

where r is the radius of the electronic orbit. Classically, r can be made arbitrarily small and so there is no lower limit to the value of E. The uncertainty principle ensures that this is not possible in quantum mechanics. Since the linear size of the atom is of order r, the uncertainty in the position of the electron is

$$\Delta r \approx r$$

According to the uncertainty principle,

$$p \approx \Delta p \approx \hbar / r$$

Therefore,

$$E = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r} \tag{5.35}$$

The system will be in the state of lowest energy at the value of r given by

$$\frac{dE}{dr} = 0$$

or $$-\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\varepsilon_0 r^2} = 0$$

or $$r = \frac{(4\pi\varepsilon_0)\hbar^2}{me^2}$$

This is same as the expression obtained for the first Bohr radius a_0 (see Equation 3.16). Its value is

$$r = a_0 = 0.53\ \text{Å}$$

Substituting in (5.35), the ground state energy of the hydrogen atom is

$$E = -\frac{me^4}{2\hbar^2 (4\pi\varepsilon_0)^2} = -\frac{\hbar^2}{2ma_0^2}$$

which is same as the Bohr's expression (see Equation 3.18). Its value is

$$E = -13.6\ \text{eV}$$

Nonexistence of Electrons Inside the Nucleus

As we know, the size of a nucleus is of the order of 10^{-14} m. Therefore, for an electron to be confined within a nucleus, the uncertainty in its position should not exceed this value. The corresponding uncertainty in the momentum of the electron would be

$$\Delta p \geq \frac{\hbar}{\Delta x} = \frac{1.054 \times 10^{-34}}{10^{-14}}$$
$$= 1.1 \times 10^{-20}\ \text{kg m/s}$$

The momentum p must be at least equal to Δp. An electron having such a large momentum has kinetic energy K much greater than its rest-mass energy m_0c^2. As such we may use the relativistic formula

$$K \approx pc$$
$$= 1.1 \times 10^{-20} \times 3 \times 10^{8}$$
$$= 3.3 \times 10^{-12}\ \text{J}$$
$$= \frac{3.3 \times 10^{-12}}{1.6 \times 10^{-13}}\ \text{MeV}$$
$$= 20.6\ \text{MeV}$$

Experiments show that the electrons emitted from nuclei in β-decay have mostly energies between 2–3 MeV. From this we conclude that electrons cannot be basic constituents of nuclei. In fact, β-decay occurs when a neutron inside the nucleus transforms into a proton, an electron and a neutrino. The electron and the neutrino are immediately ejected out of the nucleus.

Let us see how much energy an electron must possess to be confined in an atom, say hydrogen. We have

$$\Delta x \approx 5 \times 10^{-11} \text{ m}$$

Therefore,

$$p \approx \Delta p \approx \frac{\hbar}{\Delta x} = \frac{1.054 \times 10^{-34}}{5 \times 10^{-11}}$$

$$= 2.11 \times 10^{-24} \text{ kg m/s}$$

An electron having momentum of this order is nonrelativistic in behaviour. Therefore,

$$K = \frac{p^2}{2m}$$

$$= \frac{(2.11 \times 10^{-24})^2}{2 \times 9.1 \times 10^{-31}}$$

$$= 2.446 \times 10^{-18} \text{ J}$$

$$= 15.3 \text{ eV}$$

This value is reasonable.

Zero-Point Energy of a Harmonic Oscillator

We are familiar with the classical expression for the energy of a harmonic oscillator:

$$E = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

where ω is the angular frequency of oscillation. Classically, the minimum value of E is zero, which occurs when the particle is at rest ($p = 0$) at the mean position ($x = 0$). In quantum mechanics, the uncertainty principle does not allow this situation because then both position and momentum would be precisely known.

Let us assume that the particle is confined to a region of size a. Then

$$x \approx \Delta x \approx a$$

Using the exact statement of the uncertainty relation,

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

we get

$$p \approx \Delta p \approx \frac{\hbar}{2a}$$

The energy is then given by

$$E = \frac{\hbar^2}{8ma^2} + \frac{1}{2} m\omega^2 a^2 \tag{5.36}$$

For E to be minimum

$$\frac{dE}{da} = 0$$

which gives

$$a = [\hbar/2m\omega]^{1/2}$$

Substituting in (5.36), the minimum value of E is

$$E_{\min} = \frac{\hbar\omega}{2} \tag{5.37}$$

We shall see later, in a detailed study of the harmonic oscillator in chapter 9, that this is indeed the ground state energy of the harmonic oscillator. We used the exact statement of the uncertainty principle to get the right result.

This minimum energy is called the **zero-point energy**. It is clear that according to the uncertainty principle, no physical system can be completely at rest, even at absolute zero temperature. One important consequence of zero-point energy is that helium does not solidify even at very low temperatures, whereas normally a substance solidifies to form a crystal at low temperatures. Helium has a relatively shallow potential energy minimum. Moreover, being a light element, its kinetic energy is large. Therefore, it has large zero-point energy so that it remains in liquid form even at very low temperatures.

Others Applications

- Strength of nuclear Force
- Stability of atom
- Existence of neutron, proton in the nucleus
- The non existence of the electron in the nucleus