Ch-4

Spot rates:- The spot rate s_t is the rate of interest, expressed in yearly terms, charged for money held from the present $(t=0)$ time until time t. Both the interest and the original principal are paid at time t.

Hence, in particular, s_1 - is the 1-year interest rate and s_2 is the 2-year interest rate, in s_2 it will pay $A(1+s_2)^2$ at the end of 2-years.

(a) yearly, $(1+s_t)^t$

(b) m periods per year $(1+s_t/m)^{mt}$

(c) Continuous $e^{s_t t}$

Discount factor

(a) Yearly, $d_k = \frac{1}{(1+r_k)^k}$

(b) m periods per year

$$d_k = \frac{1}{(1+s_k/m)^{mk}}$$

(c) Continuous Compounding $d_t = e^{-s_t t}$

Given any cash flow stream $(x_0, x_1, \ldots, x_n)$.

Present Value $PV = x_0 + d_1 x_1 + d_2 x_2 + \ldots + d_n x_n$.

Ex:- Using spot rate, find the value of an 8% bond maturing in 10 years.

→

Year	1	2		
Spot rate	5.571	6.088		
Discount factor	0.947	0.889		
Cash flow	8	8		Price.
PV	7.58	7.11	....	97.34

Determing spot rates.

Suppose, we know s_1. Now consider 2-year bond. Suppose that bond has price P, makes coupons payments of amount C at the end of both years and has a face value F., then

$$P = \frac{c}{1+s_1} + \frac{c+F}{(1+s_2)^2}$$

there, we can solve it for s_2, working this way, by considering 3-year bonds, then 4-years bond, and so, on, we can determine $s_3, s_4, --$.

Ex:- Construction of a zero

Bond A is a 10-year bond with 10% coupon. Its price is 98.72. Bond B is 10-year bond with 8% coupon, its price is $P_B = 85.89$.

$\Rightarrow$ Consider portfolio with -0.8 unit of bond A & 1 unit of bond B. This portfolio will have a face value of 20 and a price $P_B - 0.8P_A = 6.914$. Coupon payment is zero. Then the 10-year spot rate s_{10} satisfies $(1+s_{10})^{10} P = 20$

$\Rightarrow s_{10} = 11.2\%$.

Forward Rates: forward rates are interest rates for money to be borrowed between two dates in the future, but under terms agreed upon today. Suppose s_1 and s_2 are known, If we have \$1 in a 2-year account it will grow to $\$(1+s_2)^2$ at the end of 2 years.

Alternatively, if we place \$1 in 1-year account, then it will increase \$ $(1+s_1)$. The rate f is the forward rate for money to be lent in a way s.t.

$$\$(1+s_1)(1+f) = \$(1+s_2)^2.$$

$$\Rightarrow \quad f = \frac{(1+s_2)^2}{1+s_1} - 1$$

Ex:- Suppose that the spot rates for 1 and 2 years are $s_1 = 7\%$ & $s_2 = 8\%$, then

$$f = \frac{(1+0.08)^2}{1.07} - 1 = 0.0901 = 9.01\%.$$

<u>forward rate definition</u>:- The forward rate between times t_1 and t_2 with $t_1 < t_2$ is denoted by f_{t_1,t_2}. It is the rate of interest charged for borrowing money at time t_1 which is to be repaid (with interest) at time t_2.

$$(1+s_{t_2})^{t_2} = (1+s_{t_1})^{t_1}(1+f_{t_1,t_2})^{t_2-t_1}$$

forward rate formulas:- The implied forward rate between times t_1 and $t_2 > t_1$ is the rate of interest between those times that is consistent with a given spot rate curve. Under various compounding conventions the forward rates are specified as follows.

(a) Yearly, $j > i$

$$(1+s_j)^j = (1+s_i)^i (1+f_{i,j})^{j-i}$$

Hence

$$f_{i,j} = \left[\frac{(1+s_j)^j}{(1+s_i)^i}\right]^{1/j-i} - 1.$$

(b) m periods per year

$$(1+s_j/m)^j = (1+s_i/m)^i (1+f_{i,j}/m)^{j-i}$$

Hence

$$f_{i,j} = m\left[\frac{(1+s_j/m)^j}{(1+s_i/m)^i}\right]^{1/j-1} - m$$

(c) Continuous:- for continuous compounding, the forward rates f_{t_1,t_2} are defined for all t_1 & t_2, with $t_2 > t_1$ and satisfy.

$$e^{s_{t_2} t_2} = e^{s_{t_1} t_1} e^{f_{t_1,t_2}(t_2-t_1)}.$$

Hence

$$f_{t_1,t_2} = \frac{s_{t_2} t_2 - s_{t_1} t_1}{t_2 - t_1}$$

Term Structure Explanation

There are three standard explanations for the term structure, each of which provides some important insights.

1.) Expectations theory :- first explanation is that spot rates are determined by expectations of what rates will be in the future. Spot rate curve slopes gradually upward because the market believes that the 1-year rate will most likely go up next year. (this belief is because people believe inflation will rise).

Expectation hypothesis :- Consider the forward rate $f_{1,2}$, which is the implied rate for money loaned for 1-year, a year from now. According to the expectation hypothesis, tells forward rate is exactly equal to the market expectation of what the 1-year spot rate will be next year.

2.) Liquidity Preference :- It asserts that investors usually prefer short-term fixed income securities over long-term securities. Hence to induce investors into long-term instruments, better rates must be offered for long-term instruments. For this reason, the spot rate curve rises.

3.) <u>Market Segmentation</u>:- It argues that the market for fixed-income securities is segmented by maturity dates. This viewpoint suggests that all points on the spot rate curves are mutually independent. Each is determined by the forces of supply and demand in its own market.

<u>Expectation Dynamics</u>:-

$f_{1,2} \rightarrow$ 1 year spot rate for next

$f_{13} \rightarrow$ 2 year spot rate for next year.

We begin with current spot rate curve $s_1, s_2, \ldots, s_n$ and we wish to determine next year's spot rate curve $s_1', s_2', \ldots, s_{n-1}'$. The current forward rate $f_{1,j}$ can be regarded as the expectation of what the interest rate will be next year - measured from next year's current time to a time $j-1$ years. In other words.

$$s_{j-1}' = f_{1,j} = \left[\frac{(1+s_j)^j}{1+s_1}\right]^{1/j-1} - 1$$

for $1 < j \leq n$.

The expectation process can be carried out for another step to obtain the spot rate curve for the third year, and so forth.

Example! -

	S_1	S_2	S_3	S_4
Current	6.00	6.45	6.80	7.10
forecast	6.90	7.20	7.47	

$f_{0,1}$ $f_{0,2}$ $f_{0,3}$ - - - $f_{0,n-2}$ $f_{0,n-1}$ $f_{0,n}$

$f_{1,2}$ $f_{1,3}$ $f_{1,4}$ - - $f_{1,n-1}$ $f_{1,n}$

$f_{2,3}$ $f_{2,4}$ $f_{2,5}$ - - $f_{2,n}$

$\vdots$ $\vdots$

$f_{n-2,n-1}$ $f_{n,2-n}$

$f_{n-1,n}$

Discount factor:- The symbol $d_{j,k}$ denotes the discount factor used to discount cash received at time k back to an equivalent amount of cash at time j.

$$d_{j,k} = \left[\frac{1}{1+f_{j,k}}\right]^{k-j}$$